A Generalization of the Characteristic Polynomial of a Graph

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Abstract

Given a graph G with its adjacency matrix A, the *characteristic polynomial* of G is defined as $\det(A - \lambda I)$. Two graphs which have the same characteristic polynomial are called *co-spectral*. It is known (see [2]) that there are non-isomorphic graphs which are co-spectral.

In this note we consider the following generalization of the characteristic polynomial of a graph: For a graph G with adjacency matrix A, define $A(x, y)$ as the matrix, derived from A, in which the 1s are replaced by the indeterminate x and 0s (other than the diagonals) are replaced by y. The $\mathcal{L}$-polynomial of G is defined as:

$$\mathcal{L}_G(x, y, \lambda) := \det(A(x, y) - \lambda I).$$

It follows that if two graphs have the same $\mathcal{L}$-polynomial, then they are co-spectral, as well as their complements are co-spectral. We show a (surprising) converse to this fact:

If two graphs are co-spectral, and their complements are also co-spectral, then they have the same $\mathcal{L}$-polynomial.

1 Introduction

Given two undirected simple graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ on n vertices, G and H are said to be isomorphic if there is a permutation $\pi \in S_n$ such that $\{u, v\}$ is an edge in E_1 if and only if $\{\pi(u), \pi(v)\}$ is an edge in E_2. The Graph Isomporphism problem is to decide whether two given graphs are isomorphic or not. The complexity of this problem has puzzled researchers for decades. It is not known whether there is an efficient algorithm to decide if two graphs are isomorphic or not. It is also known that it is unlikely that Graph Isomorphism is NP-Complete, as that would imply $\mathsf{PH} = \Sigma_2$. The reader is referred to the comprehensive text [4] for a detailed discussion on Graph Isomorphism.

One way to do establish that Graph Isomorphism is easy would be to find an efficiently computable graph invariant and show that it separates graphs up to their automorphism classes. It is known (see [3]) that for a graph on n vertices, there is such a set of $n^2 + 1$ polynomials, but not all of these polynomials are known to efficiently computable.

One of the earliest known efficiently computable polynomial associated to a graph is its *characteristic polynomial*. For a graph G with adjacency matrix A, the characteristic polynomial of G is defined to be $\det(A - \lambda I)$. Two graphs which have the same characteristic polynomial are called *co-spectral*. It is known (see [2]) that there are non-isomorphic graphs which are co-spectral.

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Hence, this invariant, though efficiently computable, is not powerful enough to lead to an efficient solution to the problem of Graph Isomorphism.

A few remarks are in order. There is a polynomial time algorithm to decide if G and H are isomorphic if each root of their characteristic polynomial occurs at most a constant number of times (see [1]). Indeed, if G is a random graph on n vertices, then almost surely, its characteristic polynomial has no repeated roots.

Consider the following generalization of the characteristic polynomial of a graph. For a graph G with adjacency matrix A, define $A(x, y)$ as the matrix, derived from A, in which the 1s are replaced by the indeterminate x and 0s (other than the diagonals) are replaced by y. The $\mathcal{L}$-polynomial of G is defined as:

$$\mathcal{L}_G(x, y, \lambda) := \det(A(x, y) - \lambda I).$$

This polynomial is easily seen to be efficiently computable.

The characteristic polynomial of the graph G is nothing but $\mathcal{L}_G(1, 0, \lambda)$. Moreover, the characteristic polynomial of the complement of G (denoted G_c) is $\mathcal{L}_G(0, 1, \lambda)$. Hence, the $\mathcal{L}$-polynomial *seems* to contain more information about a graph G than its characteristic polynomial. This leads to the natural question:

Which non-isomorphic graphs have the same $\mathcal{L}$-polynomial ?

From what we observed above, a necessary condition for two graphs to have the same $\mathcal{L}$-polynomial is that they are co-spectral, as well as their complements are co-spectral.

In this note, we show that this trivial necessary condition is also sufficient. More precisely, two graphs G and H have the same $\mathcal{L}$-polynomial if and only if - G and H are co-spectral, and G_c and H_c are co-spectral. Moreover, we can write down the $\mathcal{L}$-polynomial of a graph in terms of its, and its complements characteristic polynomial. It is not difficult to show (see Figure 1) that there exist non-isomorphic graphs G and H, such that G and H are co-spectral and G_c and H_c are also co-spectral.

2 The $\mathcal{L}$-polynomial of a graph

All graphs will be simple and undirected. For a graph $G = (V, E)$, let A_G denote its adjacency matrix. Usually n, which denotes the number of vertices in the graph, will be implicit in the context. Let J_n denote the $n \times n$ matrix all of whose entries are 1, and I_n be the identity matrix of order n. We will drop the subscript n when it is clear from the context. We denote the condition that G is isomorphic to H by $G \cong H$. The characteristic polynomial of a matrix A is defined to be a polynomial in λ as:

$$p_A(\lambda) := \det(A - \lambda I).$$

When A arises as the adjacency matrix of a graph G, we denote the characteristic polynomial of G as $p_G(\lambda) := p_{A_G}(\lambda)$. For a graph G, its complement G_c is defined to be the graph with the adjacency matrix $A_{G_c} := J - A_G - I$. We say $A \sim B$ if $p_A(\lambda) = p_B(\lambda)$. Two Graphs G and H (on n vertices) are said to be *co-spectral* if $p_G = p_H$. If $p_G = p_H$ and $p_{G_c} = p_{H_c}$, then call them *strongly co-spectral.*

Definition 2.1. *For a graph G, and indeterminates x, y denote the following as its $\mathcal{L}$-polynomial:*

$$\mathcal{L}_G(x, y, \lambda) := \det(xA_G + yA_{G_c} - \lambda I).$$

Notice that this polynomial is an invariant under isomorphisms.

Proposition 2.2. *If $G \cong H$, then $\mathcal{L}_G(x, y, \lambda) = \mathcal{L}_H(x, y, \lambda)$.*

Proof. If $G \cong H$, then there is a permutation matrix Π such that $\Pi^T A_G \Pi = A_H$. Also $\Pi^T A_{G_c} \Pi = A_{H_c}$. Hence, $\mathcal{L}_G(x, y, \lambda) = \det(xA_G + yA_{G_c} - \lambda I) = \det(\Pi^T(xA_G + yA_{G_c} - \lambda I)\Pi) = \det(xA_H + yA_{H_c} - \lambda I) = \mathcal{L}_H(x, y, \lambda)$. □

3 Main Result

In this section we give a characterization of the graphs with the same $\mathcal{L}$-polynomial. Formally, we prove the following theorem.

Theorem 3.1. *Let G, H be two graphs on n vertices. Then $\mathcal{L}_G = \mathcal{L}_H$ if and only if G and H are strongly co-spectral.*

Before we proceed, the following technical lemmata are needed.

Lemma 3.2. *If A, B are two $n \times n$ real matrices such that $A \sim B$, then for all $\mu \in \mathbb{R}$, $A + \mu I \sim B + \mu I$.*

Proof. Since $A \sim B$, by definition, $p_A(\lambda) = p_B(\lambda)$. Hence, for any μ, $p_A(\lambda - \mu) = p_B(\lambda - \mu)$. But $p_A(\lambda - \mu) = p_{A+\mu I}(\lambda)$ and $p_B(\lambda - \mu) = p_{B+\mu I}(\lambda)$. Hence, $A + \mu I \sim B + \mu I$. □

Lemma 3.3. *Let A, B be matrices with entries from $\mathbb{R}$. If the rank of B over $\mathbb{R}$ is at most r, then for all $\alpha \in \mathbb{R}$,*

$$\det(A + \alpha B) = c_0 + c_1\alpha + \cdots + c_r\alpha^r,$$

where $c_0, \ldots, c_r \in \mathbb{R}$.

Before we prove Lemma 3.3, we need some notation. Let $[n] := \{1, \ldots, n\}$. For a set $S \subseteq [n]$ and $i \in [n]$, define $\chi_S(i) = 1$ if $i \in S$ and $\chi_S(i) = 0$ if $i \notin S$. Write an $n \times n$ matrix A as $[\mathbf{a}_1, \ldots, \mathbf{a}_n]$, here $\mathbf{a}_i$ are the column vectors of A. The following proposition is a well known corollary of the linearity of the determinant function.

Proposition 3.4. *For two $n \times n$ matrices $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$ and $B = [\mathbf{b}_1, \ldots, \mathbf{b}_n]$ over $\mathbb{R}$,*

$$\det(A + B) = \sum_{S \subseteq [n]} \det\left([\chi_S(1)\mathbf{a}_1 + (1 - \chi_S(1))\mathbf{b}_1, \ldots, \chi_S(n)\mathbf{a}_n + (1 - \chi_S(n))\mathbf{b}_n]\right).$$

Proof. This follows from the repeated application of the following linearity of the determinant. For $B' = [\mathbf{b}_1, \mathbf{0}, \ldots, \mathbf{0}]$,

$$\det(A + B') = \det\left([\mathbf{a}_1, \ldots, \mathbf{a}_n]\right) + \det\left([\mathbf{b}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]\right).$$

□

The above sum is nothing but the sum over all 2^n matrices, each obtained by choosing a subset of columns from A and the complementary set of columns from B. Clearly if the rank of B is at most r, then any such matrix in this sum which has more than r columns from B will have determinant zero. This is exactly what is captured by Lemma 3.3 whose proof we present next.

Proof of Lemma 3.3. Write $\det(A+\alpha B)$ as promised by Proposition 3.4:

$$\sum_{j=0}^{n} \alpha^j \left(\sum_{S\subseteq[n],|S|=n-j} \det\left([\chi_S(1)\mathbf{a}_1 + (1-\chi_S(1))\mathbf{b}_1, \ldots, \chi_S(n)\mathbf{a}_n + (1-\chi_S(n))\mathbf{b}_n]\right) \right).$$

Say $\det(A+\alpha B) = c_0 + c_1\alpha + \cdots + c_n\alpha^n$. Assume on the contrary that $c_j \neq 0$, for some $j > r$. This implies that there is a set $T \subseteq [n], |T| = n-j$, such that the following matrix has non-zero determinant:

$$[\chi_T(1)\mathbf{a}_1 + (1-\chi_T(1))\mathbf{b}_1, \ldots, \chi_T(n)\mathbf{a}_n + (1-\chi_T(n))\mathbf{b}_n].$$

But this is impossible as this matrix contains at least $j \geq r+1$ columns from B and the rank of B is at most r. Hence, all such matrices must have determinant zero. □

Corollary 3.5. *If A, B are two $n\times n$ matrices over $\mathbb{R}$ such that $A \sim B$ and $J-A \sim J-B$, then for all $\gamma \in \mathbb{R}$,*

$$A + \gamma J \sim B + \gamma J.$$

Proof. By Lemma 3.3, $\det(A-\lambda I+\gamma J) = a_0(\lambda)+\gamma a_1(\lambda)$, and $\det(B-\lambda I+\gamma J) = b_0(\lambda)+\gamma b_1(\lambda)$. Here a_0, a_1, b_0, b_1 are polynomials in the indeterminate λ. Substituting $\gamma = 0$ and using the hypothesis that $A \sim B$, we get that $a_0(\lambda) = b_0(\lambda)$. Substituting $\gamma = -1$ we obtain $\det(J - A - \lambda I) = (-1)^n \det(A + \lambda I - J) = (-1)^n(a_0(-\lambda) - a_1(-\lambda))$. Similarly, $\det(J - B - \lambda I) = (-1)^n(b_0(-\lambda) - b_1(-\lambda))$. But we know that $J-A \sim J-B$, which implies that $a_0(-\lambda) - a_1(-\lambda) = b_0(-\lambda) - b_1(-\lambda)$. This, coupled with the fact that $a_0(\lambda) = b_0(\lambda)$ implies that $a_1(\lambda) = b_1(\lambda)$. Hence, we conclude that $A+\gamma J \sim B+\gamma J$, as $\det(A-\lambda I+\gamma J) = \det(B-\lambda I+\gamma J)$. □

Lemma 3.6. *For all G, H on n vertices, $\mathcal{L}_G(x,x,\lambda) = \mathcal{L}_H(x,x,\lambda)$.*

Proof. It is sufficient to show that $\mathcal{L}_G(x,x,\lambda)$ does not depend on G. Let A be the adjacency matrix of G. It follows that $\mathcal{L}_G(x,x,\lambda) = \det(xA + x(J-I-A) - \lambda I) = \det(xJ - (\lambda+x)I))$. □

Now we can proceed to the proof of the main theorem.

Proof of Theorem 3.1. The necessary direction follows trivially by substituting in $\mathcal{L}_G$ and $\mathcal{L}_H$, $(x,y) = (1,0)$ and $(0,1)$.

Now we prove the other direction. Let A, B be the adjacency matrices of G and H respectively. Since G and H are strongly co-spectral, we have that $A \sim B$ and $A - J - I \sim B - J - I$. Applying Lemma 3.2, it follows that $A - J \sim B - J$. Let $s,t,u \in \mathbb{R}$. We will show that $\mathcal{L}_G(s,t,u) = \mathcal{L}_H(s,t,u)$. By Lemma 3.6, we may assume that $s \neq t$. Apply Corollary 3.5 with $\gamma = \frac{t}{s-t}$ to obtain that $A + \frac{t}{s-t}J \sim B + \frac{t}{s-t}J$. This means that for an indeterminate λ', $p_1(\lambda') := \det\left(A + \frac{t}{s-t}J - \lambda' I\right) = \det\left(B + \frac{t}{s-t}J - \lambda' I\right) =: p_2(\lambda')$. But $(s-t)^n p_1\left(\frac{u+t}{s-t}\right) = \det((s-t)A + tJ - (u+t)I) = \mathcal{L}_G(s,t,u)$. Similarly $(s-t)^n p_2\left(\frac{u+t}{s-t}\right) = \mathcal{L}_H(s,t,u)$. But $p_1 = p_2$ and hence $\mathcal{L}_G(s,t,u) = \mathcal{L}_H(s,t,u)$. Hence, the proof is complete. □

The proof above suggests the following self-evident corollary. This gives us an explicit formula for the $\mathcal{L}$-polynomial of a graph.

Corollary 3.7. *Given a graph G, on n vertices. Let $p := p_{A_G}$ and $\overline{p} := p_{A_{G_c}}$. Then*

$$\mathcal{L}_G(x,y,\lambda) = (x-y)^{n-1}\left[xp\left(\frac{\lambda+y}{x-y}\right) + y\overline{p}\left(-\frac{\lambda+x}{x-y}\right)\right].$$

Here the right hand side is to be interpreted as a polynomial rather than a rational function.

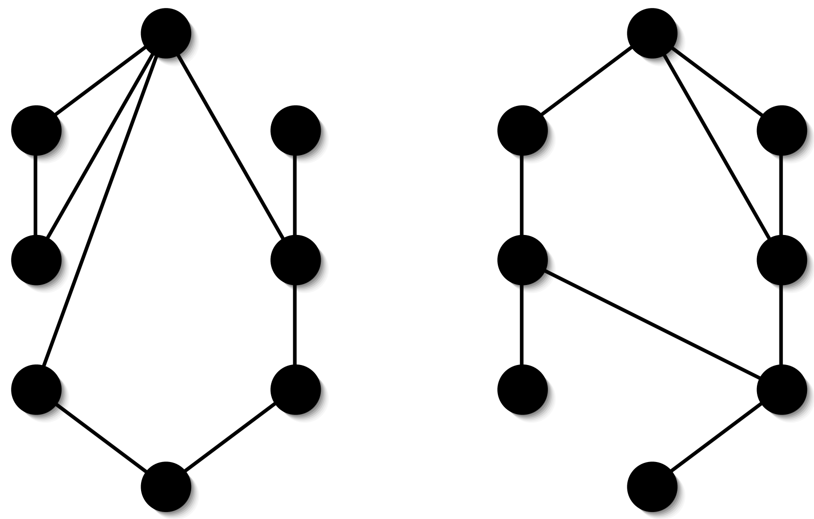

Figure 1: Non-isomorphic strongly co-spectral graphs

4 Non-isomorphic strongly co-spectral graphs

The characteristic polynomial of the graphs in Figure 1 is

$$1 - 4\lambda - 15\lambda^2 + 6\lambda^3 + 22\lambda^4 - 2\lambda^5 - 9\lambda^6 + \lambda^8.$$

The characteristic polynomial of their complements is

$$-16\lambda - 4\lambda^2 + 48\lambda^3 + 23\lambda^4 - 30\lambda^5 - 19\lambda^6 + \lambda^8.$$

References

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