The Unique Games Conjecture, Integrality Gap for Cut Problems and Embeddability of Negative Type Metrics into ℓ_1

[Extended Abstract]

Subhash A. Khot
College of Computing, Georgia Tech
Atlanta GA 30332
khot@cc.gatech.edu

Nisheeth K. Vishnoi
IBM India Research Lab
Block-1 IIT Delhi, New Delhi 110016
nvishnoi@in.ibm.com

Abstract

In this paper we disprove the following conjecture due to Goemans [16] and Linial [24] (also see [5, 26]): "Every negative type metric embeds into ℓ_1 with constant distortion." We show that for every $\delta > 0$, and for large enough n, there is an n-point negative type metric which requires distortion at-least $(\log\log n)^{1/6-\delta}$ to embed into ℓ_1.

*Surprisingly, our construction is inspired by the Unique Games Conjecture (*UGC*) of Khot [19], establishing a previously unsuspected connection between* PCP*s and the theory of metric embeddings. We first prove that the* UGC *implies super-constant hardness results for (non-uniform)* SPARSEST CUT *and* MINIMUM UNCUT *problems. It is already known that the* UGC *also implies an optimal hardness result for* MAXIMUM CUT *[20].*

Though these hardness results depend on the UGC*, the integrality gap instances rely "only" on the* PCP *reductions for the respective problems. Towards this, we first construct an integrality gap instance for a natural* SDP *relaxation of* UNIQUE GAMES*. Then, we "simulate" the* PCP *reduction and "translate" the integrality gap instance of* UNIQUE GAMES *to integrality gap instances for the respective cut problems! This enables us to prove a $(\log\log n)^{1/6-\delta}$ integrality gap for (non-uniform)* SPARSEST CUT *and* MINIMUM UNCUT*, and an optimal integrality gap for* MAXIMUM CUT*. All our* SDP *solutions satisfy the so-called "triangle inequality" constraints. This also shows, for the first time, that the triangle inequality constraints do not add any power to the Goemans-Williamson's* SDP *relaxation of* MAXIMUM CUT.

The integrality gap for SPARSEST CUT *immediately implies a lower bound for embedding negative type metrics into ℓ_1. It also disproves the non-uniform version of Arora, Rao and Vazirani's Conjecture [5], asserting that the integrality gap of the* SPARSEST CUT SDP*, with the triangle inequality constraints, is bounded from above by a constant.*

1. Introduction

In recent years, the theory of metric embeddings has played an increasing role in algorithm design. Best approximation algorithms for several NP-hard problems rely on techniques (and theorems) used to embed one metric space into another with *low distortion*.

Bourgain [7] showed that every n-point metric embeds into ℓ_1 (in fact into ℓ_2) with distortion $O(\log n)$. Independently, Aumann and Rabani [6] and Linial, London and Rabinovich [25] gave a striking application of Bourgain's Theorem: An $O(\log n)$ approximation algorithm for SPARSEST CUT. The approximation ratio is exactly the distortion incurred in Bourgain's Theorem. This gave an alternate approach to the seminal work of Leighton and Rao [23], who obtained an $O(\log n)$ approximation algorithm for SPARSEST CUT via a LP-relaxation based on muticommodity flows. It is well-known that an $f(n)$ factor algorithm for SPARSEST CUT can be used iteratively to design an $O(f(n))$ factor algorithm for BALANCED SEPARATOR: Given a graph that has a $(\frac{1}{2}, \frac{1}{2})$-partition cutting an α fraction of the edges, the algorithm produces a $(\frac{1}{3}, \frac{2}{3})$-partition that cuts at-most $O(f(n)\alpha)$ fraction of the edges. Such partitioning algorithms are very useful as sub-routines in designing graph theoretic algorithms via the divide-and-conquer paradigm.

The results of [6, 25] are based on the *metric* LP *relaxation* of SPARSEST CUT. Given an instance $G(V, E)$ of SPARSEST CUT, let d_G be the n-point metric obtained as a solution to this LP. The metric d_G is then embedded into ℓ_1 via Bourgain's Theorem. Since ℓ_1 metrics are non-negative linear combinations of cut metrics, an embedding into ℓ_1 essentially gives the desired sparse cut (up to an $O(\log n)$ approximation factor). Subsequent to this result, it was realized that one could write an SDP relaxation of SPARSEST CUT, and enforce an additional condition, that the metric d_G belong to a special class of metrics, called the *negative type metrics* (denoted by ℓ_2^2). Clearly, if ℓ_2^2 embeds into ℓ_1 with distortion $g(n)$, then one would get a $g(n)$ approximation to SPARSEST CUT.[1]

The results of [6, 25] led to the conjecture that ℓ_2^2 embeds into ℓ_1 with distortion C_{neg}, for some absolute constant C_{neg}. This conjecture has been attributed to Goemans [16] and Linial [24], see [5, 26]. This conjecture, which we will henceforth refer to as the $(\ell_2^2, \ell_1, O(1))$-Conjecture, if true, would have had tremendous algorithmic applications (apart from being an important mathematical result). Several problems, specifically cut problems (see [11]), can be formulated as optimization problems over the class of ℓ_1 metrics, and optimization over ℓ_1 is an NP-hard problem in general. However, one can optimize over ℓ_2^2 metrics in polynomial time via SDPs (and $\ell_1 \subseteq \ell_2^2$). Hence, if ℓ_2^2 was embeddable into ℓ_1 with constant distortion, one would get a computationally efficient approximation to ℓ_1 metrics.

However, no better embedding of ℓ_2^2 into ℓ_1, other than Bourgain's $O(\log n)$ embedding (that works for all metrics), was known until recently. A breakthrough result of Arora, Rao and Vazirani (ARV) [5] gave an $O(\sqrt{\log n})$ approximation to (uniform) SPARSEST CUT by showing that the integrality gap of the SDP relaxation is $O(\sqrt{\log n})$ (see also [28] for an alternate perspective on ARV). Subsequently, ARV techniques were used by Chawla, Gupta and Räcke [9] to give an $O(\log^{3/4} n)$ distortion embedding of ℓ_2^2 metrics into ℓ_2, and hence, into ℓ_1. This result was further improved to $O(\sqrt{\log n} \log \log n)$ by Arora, Lee, and Naor [3]. The latter paper implies, in particular, that every n-point ℓ_1 metric embeds into ℓ_2 with distortion $O(\sqrt{\log n} \log \log n)$, almost matching decades old $\Omega(\sqrt{\log n})$ lower bound due to Enflo [12]. Techniques from ARV have also been applied, to obtain $O(\sqrt{\log n})$ approximation to MINIMUM UNCUT and related problems [1], to VERTEX SEPARATOR [13], and to obtain a $2 - O(\frac{1}{\sqrt{\log n}})$ approximation to VERTEX COVER [18]. It was conjectured in the ARV paper, that the integrality gap of the SDP relaxation of SPARSEST CUT is bounded from above by an absolute constant (they make this conjecture only for the uniform version, and the $(\ell_2^2, \ell_1, O(1))$-Conjecture implies it also for the non-uniform version). Thus, if the $(\ell_2^2, \ell_1, O(1))$-Conjecture and/or the ARV-Conjecture were true, one would potentially get a constant factor approximation to a host of problems, and perhaps, an algorithm for VERTEX COVER with an approximation factor better than 2! Clearly, it is an important open problem to prove or disprove the $(\ell_2^2, \ell_1, O(1))$-Conjecture and/or the ARV-Conjecture. The main result in this paper is a disproval of the $(\ell_2^2, \ell_1, O(1))$-Conjecture and a disproval of the non-uniform version of the ARV-Conjecture, see Conjecture A.15.[2] The disprovals follow from the construction of a super-constant integrality gap for the non-uniform version of BALANCED SEPARATOR (which implies the same gap for the non-uniform version of SPARSEST CUT). We also obtain integrality gap instances for MAXIMUM CUT and MINIMUM UNCUT. In the following sections, we describe our results in detail and present an overview of our ℓ_2^2 versus ℓ_1 lower bound. Due to the lack of space, the only technical details we are able to provide in this version of the paper are: The basic setup, which relates cuts and metrics, in Appendix A, and the overall approach for the disproval of $(\ell_2^2, \ell_1, O(1))$-Conjecture in Appendix B. The full version of this paper will have all the details.

[1] Algorithms based on metric embeddings (typically) work for the *non-uniform* version of SPARSEST CUT, which is more general. The Leighton-Rao algorithm worked only for the uniform version.

[2] We believe that even the uniform version of the ARV-Conjecture is false.

2. Our Results

2.1. The Disproval of $(\ell_2^2, \ell_1, O(1))$-Conjecture

We prove the following theorem which follows from the integrality gap construction for non-uniform BALANCED SEPARATOR. See Section A for definitions and basic facts.

Theorem 2.1 *For every $\delta > 0$ and for all sufficiently large n, there is an n-point ℓ_2^2 metric which cannot be embedded into ℓ_1 with distortion less than $(\log \log n)^{1/6-\delta}$.*

Remark 2.2 *One of the crucial ingredients for obtaining the lower bound of $(\log \log n)^{1/6-\delta}$ in Theorems 2.1 and 2.3 is Bourgain's Junta Theorem [8]. A recent improvement of this theorem due to Mossel* et al. *[27] improves both of our lower bounds to $(\log \log n)^{1/4-\delta}$.*

2.2. Integrality Gap Instances for Cut Problems

SPARSEST CUT and BALANCED SEPARATOR (non-uniform versions), as well as MAXIMUM CUT and MINIMUM UNCUT are defined in Section A.4. Natural SDP relaxations for these problems are also described there. All the SDPs include the so-called *triangle inequality* constraints: For every triple of vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in the SDP solution, $\|\mathbf{u} - \mathbf{v}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 \geq \|\mathbf{u} - \mathbf{w}\|^2$. Note that these constraints are always satisfied by the *integral* solutions, i.e., $+1, -1$ valued solutions. We prove the following two theorems:

Theorem 2.3 SPARSEST CUT, BALANCED SEPARATOR *(non-uniform versions of both) and* MINIMUM UNCUT *have an integrality gap of at-least $(\log \log n)^{1/6-\delta}$, where*

$\delta > 0$ *is arbitrary. The integrality gap holds for standard* SDP*s with triangle inequality constraints.*

Theorem 2.4 *Let* α_{GW} *(* ≈ 0.878*) be the approximation ratio obtained by Goemans-Williamson's algorithm for* MAXIMUM CUT *[17]. For every* $\delta > 0$*, the Goemans-Williamson's* SDP *has an integrality gap of at-least* $\alpha_{\mathsf{GW}} + \delta$*, even after including the triangle inequality constraints.*

This theorem relies on a Fourier analytic result called *Majority is Stablest Theorem* due to Mossel *et al.* [27].

We note that without the triangle inequality constraints, Feige and Schechtman [15] already showed an $\alpha_{\mathsf{GW}} + \delta$ integrality gap. One more advantage of our result is that it is an explicit construction, where as Feige and Schechtman's construction is randomized (they need to pick random points on the unit sphere). Our result shows that adding the triangle inequality constraints does not add any power to the Goemans-Williamson's SDP. This nicely complements the result of Khot *et al.* [20], where it is shown that, assuming the Unique Games Conjecture (UGC), it is NP-hard to approximate MAXIMUM CUT within a factor better than $\alpha_{\mathsf{GW}} + \delta$.

2.3. Hardness Results for SPARSEST CUT and BALANCED SEPARATOR Assuming the UGC

Our starting point is the hardness of approximation results for cut problems assuming the UGC (see Section C for the statement of the conjecture). We prove the following result:

Theorem 2.5 *Assuming the* UGC*,* SPARSEST CUT *and* BALANCED SEPARATOR *(non-uniform versions) are* NP*-hard to approximate within any constant factor.*

This particular result was also proved[3] by Chawla *et al.* [10]. Similar result for MINIMUM UNCUT is implicit in [19], where the author formulated the UGC and proved the hardness of approximating MIN-2SAT-DELETION. As mentioned before, Khot *et al.* [20] proved that the UGC implies $\alpha_{\mathsf{GW}} + \delta$ hardness result for MAXIMUM CUT. As an aside, we note that the UGC also implies optimal $2 - \delta$ hardness result for VERTEX COVER, as shown in [22].

Therefore, assuming the UGC, all of the above problems are NP-hard to approximate within respective factors, and hence, the corresponding integrality gap examples must exist (unless P=NP). In particular, if the UGC is true, then the $(\ell_2^2, \ell_1, O(1))$-Conjecture is false. This is a rather peculiar situation, because the UGC is still unproven, and may very well be false. Nevertheless, we are able to disprove the $(\ell_2^2, \ell_1, O(1))$-Conjecture *unconditionally* (which may be taken as an argument supporting the UGC). Indeed, the UGC plays a crucial role in our disproval. Let us outline the basic approach we take. First, we build an integrality gap instance for a natural SDP relaxation of UNIQUE GAMES (see Figure 5). Surprisingly, we are then able to *translate* this integrality gap instance into an integrality gap instance of SPARSEST CUT, BALANCED SEPARATOR, MAXIMUM CUT and MINIMUM UNCUT. This translation *mimics* the PCP reduction from the UGC to these problems (note that the same reduction also proves hardness results assuming the UGC)! We believe that this novel approach will have several applications in the future. Already, inspired by our work, Khot and Naor [21] have proved several non-embeddability results (e.g. Edit Distance into ℓ_1), and Arora *et al.* [2] have constructed integrality gap instances for the MAXQP problem.

[3] We would like to stress that our work was completely independent, and no part of our work was influenced by their paper.

2.4. Integrality Gap Instance for the UNIQUE GAMES SDP Relaxation

As mentioned above, we construct an integrality gap instance for a natural SDP relaxation of UNIQUE GAMES (see Figure 5). Here, we choose to provide an informal description of this construction (the reader should be able to understand this construction without even looking at the SDP formulation).

Theorem 2.6 *(Informal statement) Let* N *be an integer and* $\eta > 0$ *be a parameter (think of* N *as large and* η *as very tiny). There is a graph* $G(V, E)$ *of size* $2^N/N$ *with the following properties: Every vertex* $u \in V$ *is assigned a set of unit vectors* $B(u) := \{\mathbf{u}_1, \ldots, \mathbf{u}_N\}$ *that form an orthonormal basis for the space* $\mathbb{R}^N$*. Further,*

1. *For every edge* $e = (u, v) \in E$*, the set of vectors* $B(u)$ *and* $B(v)$ *are almost the same upto some small perturbation. To be precise, there is a permutation* $\pi_e : [N] \mapsto [N]$*, such that* $\forall\ 1 \le i \le N,\ \langle \mathbf{u}_{\pi_e(i)}, \mathbf{v}_i \rangle \ge 1 - \eta$*. In other words, for every edge* $(u, v) \in E$*, the basis* $B(u)$ *moves "smoothly/continuously" to the basis* $B(v)$*.*

2. *For any* labeling $\lambda : V \mapsto [N]$*, i.e., assignment of an integer* $\lambda(u) \in [N]$ *to every* $u \in V$*, for at-least* $1 - \frac{1}{N^\eta}$ *fraction of the edges* $e = (u, v) \in E$*, we have* $\pi_e(\lambda(u)) \ne \lambda(v)$*. In other words, no matter how we choose to assign a vector* $\mathbf{u}_{\lambda(u)} \in B(u)$ *for every vertex* $u \in V$*, the movement from* $\mathbf{u}_{\lambda(u)}$ *to* $\mathbf{v}_{\lambda(v)}$ *is "discontinuous" for almost all edges* $(u, v) \in E$*.*

3. *All vectors in* $\cup_{u \in V} B(u)$ *have co-ordinates in the set* $\{\frac{1}{\sqrt{N}}, \frac{-1}{\sqrt{N}}\}$*, and hence, any three of them satisfy the triangle inequality constraint.*

The construction is rather non-intuitive: One can walk on the graph G by changing the basis $B(u)$ continuously, but as soon as one picks a *representative vector* for each basis, the motion becomes discontinuous almost everywhere! Of course, one can pick these representatives in a continuous fashion for any small enough local sub-graph of G, but there is no way to pick representatives in a global fashion. This construction eventually leads us to a ℓ_2^2 metric which, roughly speaking, is locally ℓ_1-embeddable, but globally, it requires super-constant distortion to embed into ℓ_1 (such local versus global phenomenon has also been observed by Arora *et al.* [4]).

3 Difficulty in Proving ℓ_2^2 vs. ℓ_1 Lower Bound

In this section, we describe the difficulties in constructing ℓ_2^2 metrics that do not embed well into ℓ_1. This might partly explain why one needs an unusual construction as the one in this paper. Our discussion here is informal, without precise statements or claims.

Difficulty in constructing ℓ_2^2 metrics: To the best of our knowledge, no natural families of ℓ_2^2 metrics are known other than the Hamming metric on $\{-1,1\}^k$. The Hamming metric is an ℓ_1 metric, and hence, not useful for the purposes of obtaining ℓ_1 lower bounds. Certain ℓ_2^2 metrics can be constructed via Fourier analysis, and one can also construct some by solving SDPs explicitly. The former approach has a drawback that metrics obtained via Fourier methods typically embed into ℓ_1 isometrically. The latter approach has limited scope, since one can only hope to solve SDPs of moderate size. Feige and Schechtman [15] show that selecting an appropriate number of points from the unit sphere gives a ℓ_2^2 metric. However, in this case, most pairs of points have distance $\Omega(1)$ and hence, the metric is likely to be ℓ_1-embeddable with low distortion.

Difficulty in proving ℓ_1 lower bounds: To the best of our knowledge, there is no standard technique to prove a lower bound for embedding a metric into ℓ_1. The only interesting (super-constant) lower bound that we know is due to [6, 25], where it is shown that the shortest path metric on a constant degree expander requires $\Omega(\log n)$ distortion to embed into ℓ_1.

General theorems regarding group norms: A *group norm* is a distance function $d(\cdot,\cdot)$ on a group $(G,\circ)$, such that $d(x,y)$ depends only on the group difference $x \circ y^{-1}$. Using Fourier methods, it is possible to construct group norms that are ℓ_2^2 metrics. However, it is known that any group norm on $\mathbb{R}^k$, or on any group of characteristic 2, is isometrically ℓ_1-embeddable (see [11]). It is also known (among the experts in this area) that such a result holds for every abelian group. Therefore, any approach, just via group norms would be unlikely to succeed, as long as the underlying group is abelian. (But, only in the abelian case, the Fourier methods work well.)

The best known lower bounds for the ℓ_2^2 versus ℓ_1 question were due to Vempala ($\frac{10}{9}$ for a metric obtained by a computer search), and Goemans (1.024 for a metric based on the Leech Lattice), see [29]. Thus, it appeared that an entirely new approach was needed to resolve the $(\ell_2^2, \ell_1, O(1))$-Conjecture. In this paper, we present an approach based on tools from complexity theory, namely, the UGC, PCPs, and Fourier analysis of boolean functions. Interestingly, Fourier analysis is used both to construct the ℓ_2^2 metric, as well as, to prove the ℓ_1 lower bound.

4. Overview of Our ℓ_2^2 vs. ℓ_1 Lower Bound

In this section, we present a high level idea of our ℓ_2^2 versus ℓ_1 lower bound (see Theorem 2.1). Given the construction of Theorem 2.6, it is fairly straight-forward to describe the candidate ℓ_2^2 metric: Let $G(V,E)$ be the graph, and $B(u)$ be the orthonormal basis for $\mathbb{R}^N$ for every $u \in V$, as in Theorem 2.6. Fix $s = 4$. For $u \in V$ and $\mathbf{x} = (x_1,\ldots,x_N) \in \{-1,1\}^N$, define the vector $\mathbf{V}_{u,s,\mathbf{x}}$ as follows:

$$\mathbf{V}_{u,s,\mathbf{x}} := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i \mathbf{u}_i^{\otimes 2s} \tag{1}$$

Note that since $B(u) = \{\mathbf{u}_1,\ldots,\mathbf{u}_N\}$ is an orthonormal basis for $\mathbb{R}^N$, every $\mathbf{V}_{u,s,\mathbf{x}}$ is a unit vector. Fix t to be a large odd integer, for instance $2^{240}+1$, and consider the set of unit vectors $\mathcal{S} = \{\mathbf{V}_{u,s,\mathbf{x}}^{\otimes t} \mid u \in V,\ \mathbf{x} \in \{-1,1\}^N\}$. Using, essentially, the fact that the vectors in $\cup_{u\in V} B(u)$ are a *good* solution to the SDP relaxation of UNIQUE GAMES, we are able to show that every triple of vectors in $\mathcal{S}$ satisfy the triangle inequality constraint and, hence, $\mathcal{S}$ defines a ℓ_2^2 metric. One can also directly show that this ℓ_2^2 metric does not embed into ℓ_1 with distortion less than $(\log N)^{1/6-\delta}$.

However, we choose to present our construction in a different and a quite indirect way. The (lengthy) presentation goes through the Unique Games Conjecture, and the PCP reduction from UNIQUE GAMES integrality gap instance to BALANCED SEPARATOR. Hopefully, our presentation will bring out the intuition as to why and how we came up with the above set of vectors, which happened to define a ℓ_2^2 metric. At the end, the reader will recognize that the idea of taking all $+/-$ linear combinations of vectors in $B(u)$ (as in Equation (1)) is directly inspired by the PCP reduction. Also, the proof of the ℓ_1 lower bound will be hidden inside the *soundness analysis* of the PCP!

The overall construction can be divided into three steps:

1. A PCP reduction from UNIQUE GAMES to BALANCED SEPARATOR.

2. Constructing an integrality gap instance for a natural SDP relaxation of UNIQUE GAMES.

3. Combining these two to construct an integrality gap instance of BALANCED SEPARATOR. This also gives a ℓ_2^2 metric that needs $(\log \log n)^{1/6-\delta}$ distortion to embed into ℓ_1.

We present an overview of each of these steps in three separate sections. Before we do that, let us summarize the precise notion of an integrality gap instance of BALANCED SEPARATOR. To keep things simple in this exposition, we will pretend as if our construction works for the uniform version of BALANCED SEPARATOR as well. (Actually, it doesn't. We have to work with the non-uniform version and it complicates things a little.)

4.1. SDP Relaxation of BALANCED SEPARATOR

Given a graph $G'(V', E')$, BALANCED SEPARATOR asks for a $(\frac{1}{2}, \frac{1}{2})$-partition of V' that cuts as few edges as possible. (However, the algorithm is allowed to output a roughly balanced partition, say $(\frac{1}{4}, \frac{3}{4})$-partition.) Following is an SDP relaxation of BALANCED SEPARATOR:

$$\text{Minimize} \quad \frac{1}{|E'|} \sum_{e'=\{i,j\} \in E'} \frac{1}{4} \|\mathbf{v}_i - \mathbf{v}_j\|^2 \tag{2}$$

Subject to

$$\forall\, i \in V' \qquad \|\mathbf{v}_i\|^2 = 1 \tag{3}$$

$$\forall\, i, j, l \in V' \quad \|\mathbf{v}_i - \mathbf{v}_j\|^2 + \|\mathbf{v}_j - \mathbf{v}_l\|^2 \geq \|\mathbf{v}_i - \mathbf{v}_l\|^2 \tag{4}$$

$$\textstyle\sum_{i<j} \|\mathbf{v}_i - \mathbf{v}_j\|^2 \geq |V'|^2 \tag{5}$$

Figure 1. SDP relaxation of the uniform version of BALANCED SEPARATOR

Note that a $\{+1, -1\}$-valued solution represents a true partition, and hence, this is an SDP relaxation. Constraint (4) is the triangle inequality constraint and Constraint (5) stipulates that the partition be balanced. The notion of integrality gap is summarized in the following definition:

Definition 4.1 *An integrality gap instance of* BALANCED SEPARATOR *is a graph $G'(V', E')$ and an assignment of unit vectors $i \mapsto \mathbf{v}_i$ to its vertices such that:*

- *Every almost balanced partition (say $(\frac{1}{4}, \frac{3}{4})$-partition; the choice is arbitrary) of V' cuts at-least α fraction of edges.*

- *The set of vectors $\{\mathbf{v}_i |\ i \in V'\}$ satisfy* (3)-(5)*, and the* SDP *objective value in Equation* (2) *is at-most γ.*

The integrality gap is defined to be α/γ. (Thus, we desire that $\gamma \ll \alpha$.)

The next three sections describe the three steps involved in constructing an integrality gap instance of BALANCED SEPARATOR. Once that is done, it follows from a folklore result that the resulting ℓ_2^2 metric (defined by vectors $\{\mathbf{v}_i |\ i \in V'\}$) requires distortion at-least $\Omega(\alpha/\gamma)$ to embed into ℓ_1. This would prove Theorem 2.1 with an appropriate choice of parameters.

4.2. The PCP Reduction from UNIQUE GAMES to BALANCED SEPARATOR

An instance $\mathcal{U}(G(V, E), [N], \{\pi_e\}_{e \in E})$ of UNIQUE GAMES consists of a graph $G(V, E)$ and permutations $\pi_e : [N] \mapsto [N]$ for every edge $e = (u, v) \in E$. The goal is to find a *labeling* $\lambda : V \mapsto [N]$ that *satisfies* as many edges as possible. An edge $e = (u, v)$ is satisfied if $\pi_e(\lambda(u)) = \lambda(v)$. Let $\mathsf{OPT}(\mathcal{U})$ denote the maximum fraction of edges satisfied by any labeling.

UGC *(Informal Statement): It is* NP*-hard to decide whether an instance $\mathcal{U}$ of* UNIQUE GAMES *has $\mathsf{OPT}(\mathcal{U}) \geq 1-\eta$ (*YES *instance) or $\mathsf{OPT}(\mathcal{U}) \leq \zeta$ (*NO *instance), where $\eta, \zeta > 0$ can be made arbitrarily small by choosing N to be a sufficiently large constant.*

It is possible to construct an instance of BALANCED SEPARATOR $G'_\varepsilon(V', E')$ from an instance $\mathcal{U}(G(V, E), [N], \{\pi_e\}_{e \in E})$ of UNIQUE GAMES. We describe only the high level idea here. The construction is parameterized by $\varepsilon > 0$. The graph G'_ε has a block of 2^N vertices for every $u \in V$. This block contains one vertex for every point in the boolean hypercube $\{-1, 1\}^N$. Denote the set of these vertices by $V'[u]$. More precisely,

$$V'[u] := \{(u, \mathbf{x}) \mid \mathbf{x} \in \{-1, 1\}^N\}$$

We let $V' := \cup_{u \in V} V'[u]$. For every edge $e = (u, v) \in E$, the graph G'_ε has edges between the blocks $V'[u]$ and $V'[v]$. These edges are supposed to capture the constraint that the labels of u and v are consistent (i.e. $\pi_e(\lambda(u)) = \lambda(v)$). Roughly speaking, a vertex $(u, \mathbf{x}) \in V'[u]$ is connected to a vertex $(v, \mathbf{y}) \in V'[v]$ if and only if, after identifying the co-ordinates in $[N]$ via the permutation π_e, the Hamming distance between the bit-strings $\mathbf{x}$ and $\mathbf{y}$ is at-most εN.

This reduction has the following two properties:

Theorem 4.2 *(*PCP *reduction: Informal statement)*

1. *(Completeness/*YES *case): If $\mathsf{OPT}(\mathcal{U}) \geq 1 - \eta$, then the graph G'_ε has a $(\frac{1}{2}, \frac{1}{2})$-partition that cuts at-most $\eta + \varepsilon$ fraction of its edges.*

2. *(Soundness/NO Case): If* $\mathsf{OPT}(\mathcal{U}) \le 2^{-O(1/\varepsilon^2)}$*, then every* $(\frac{1}{4}, \frac{3}{4})$*-partition of* G'_ε *cuts at-least* $\sqrt{\varepsilon}$ *fraction of its edges.*

Remark 4.3 *We were imprecise on two counts: (1) The soundness property holds only for those partitions that partition a constant fraction of the blocks* $V'[u]$ *in a roughly balanced way. We call such partitions* piecewise balanced*. This is where the issue of uniform versus non-uniform version of* BALANCED SEPARATOR *arises. (2) For the soundness property, we can only claim that every piecewise balanced partition cuts at least* ε^t *fraction of edges, where any* $t > \frac{1}{2}$ *can be chosen in advance. Instead, we write* $\sqrt{\varepsilon}$ *for the simplicity of notation.*

4.3. Integrality Gap Instance for the UNIQUE GAMES SDP Relaxation

This has already been described in Theorem 2.6. The graph $G(V, E)$ therein along with the ortho-normal basis $B(u)$, for every $u \in V$, can be used to construct an instance $\mathcal{U}(G(V, E), [N], \{\pi_e\}_{e\in E})$ of UNIQUE GAMES. For every edge $e = (u, v) \in E$, we have an (unambiguously defined) permutation $\pi_e : [N] \mapsto [N]$, where $\langle \mathbf{u}_{\pi_e(i)}, \mathbf{v}_i\rangle \ge 1 - \eta$, for all $1 \le i \le N$.

Theorem 2.6 implies that $\mathsf{OPT}(\mathcal{U}) \le \frac{1}{N^\eta}$. On the other hand, the fact that for every edge $e = (u, v)$, the bases $B(u)$ and $B(v)$ are very close to each other means that the SDP objective value for $\mathcal{U}$ is at-least $1 - \eta$. (Formally, the SDP objective value is defined to be $E_{e=(u,v)\in E}\left[\frac{1}{N}\sum_{i=1}^N \langle \mathbf{u}_{\pi_e(i)}, \mathbf{v}_i\rangle\right]$.)

Thus, we have a concrete instance of UNIQUE GAMES with optimum at most $\frac{1}{N^\eta} = o(1)$, and which has an SDP solution with objective value at-least $1 - \eta$. This is what an integrality gap example means: The SDP solution *cheats* in an unfair way!

4.4. Integrality Gap Instance for the BALANCED SEPARATOR SDP Relaxation

Now we combine the two modules described above. We take the instance $\mathcal{U}(G(V, E), [N], \{\pi_e\}_{e\in E})$ as above, and run the PCP reduction on it. This gives us an instance $G'(V', E')$ of BALANCED SEPARATOR. We show that this is an integrality gap instance in the sense of Defintion 4.1.

Since $\mathcal{U}$ is a NO instance of UNIQUE GAMES (i.e. $\mathsf{OPT}(\mathcal{U}) = o(1)$), Theorem 4.2 implies that every (piecewise) balanced partition of G' must cut at-least $\sqrt{\varepsilon}$ fraction of the edges. We need to have $1/N^\eta \le 2^{-O(1/\varepsilon^2)}$ for this to hold.

On the other hand, we can construct an SDP solution for the BALANCED SEPARATOR instance which has an objective value of at-most $O(\eta + \varepsilon)$. Note that a typical vertex of G' is $(u, \mathbf{x})$, where $u \in V$ and $\mathbf{x} \in \{-1, 1\}^N$. To this vertex, we attach the unit vector $\mathbf{V}_{u,s,\mathbf{x}}^{\otimes t}$ (for $s = 4, t = 2^{240} + 1$), where

$$\mathbf{V}_{u,s,\mathbf{x}} := \frac{1}{\sqrt{N}}\sum_{i=1}^N x_i \mathbf{u}_i^{\otimes 2s}.$$

It can be shown that the set of vectors $\{\mathbf{V}_{u,s,\mathbf{x}}^{\otimes t} \mid u \in V,\ \mathbf{x} \in \{-1, 1\}^N\}$ satisfy the triangle inequality constraint, and hence, defines a ℓ_2^2 metric. Vectors $\mathbf{V}_{u,s,\mathbf{x}}^{\otimes t}$ and $\mathbf{V}_{u,s,-\mathbf{x}}^{\otimes t}$ are antipodes of each other, and hence, the SDP Constraint (5) is also satisfied. Finally, we show that the SDP objective value (Expression (2)) is $O(\eta + \varepsilon)$. It suffices to show that for every edge $((u, \mathbf{x}), (v, \mathbf{y}))$ in $G'(V', E')$, we have

$$\left\langle \mathbf{V}_{u,s,\mathbf{x}}^{\otimes t}, \mathbf{V}_{v,s,\mathbf{y}}^{\otimes t}\right\rangle \ \ge\ 1 - O(st(\eta + \varepsilon)).$$

This holds because, whenever $((u, \mathbf{x}), (v, \mathbf{y}))$ is an edge of G', we have (after identifying the indices via the permutation $\pi_e : [N] \mapsto [N]$): (a) $\langle \mathbf{u}_{\pi_e(i)}, \mathbf{v}_i\rangle \ge 1 - \eta$, for all $1 \le i \le N$. (b) The Hamming distance between $\mathbf{x}$ and $\mathbf{y}$ is at-most εN.

4.5. Quantitative Parameters

It follows from above discussion (see also Definition 4.1) that the integrality gap for BALANCED SEPARATOR is $\Omega(1/\sqrt{\varepsilon})$ provided that $\eta \approx \varepsilon$, and $N^\eta > 2^{O(1/\varepsilon^2)}$. We can choose $\eta \approx \varepsilon \approx (\log N)^{-1/3}$. Since the size of the graph G' is at-most $n = 2^{2N}$, we see that the integrality gap is $\approx (\log\log n)^{1/6}$ as desired.

4.6. Proving the Triangle Inequality

As mentioned above, one can show that the set of vectors $\{\mathbf{V}_{u,s,\mathbf{x}}^{\otimes t} \mid u \in V,\ \mathbf{x} \in \{-1, 1\}^N\}$ satisfy the triangle inequality constraints. This is the most technical part of the paper, but we would like to stress that this is where the "magic" happens. In our construction, all vectors in $\cup_{u\in V} B(u)$ happen to be points of the hypercube $\{-1, 1\}^N$ (upto a normalizing factor of $1/\sqrt{N}$), and therefore, they define an ℓ_1 metric. The apparently *outlandish* operation of taking their $+/-$ combinations combined with tensoring, miraculously leads to a metric that is (ℓ_2^2 and) non-ℓ_1-embeddable.

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A. Preliminaries

A.1. Metric Spaces

Definition A.1 *(X, d) is a metric space, or d is a metric on X if: (1) For all $x \in X$, $d(x, x) = 0$. (2) For all $x, y \in X, x \neq y$ $d(x, y) > 0$. (3) For all $x, y \in X$, $d(x, y) = d(y, x)$. (4) For all $x, y, z \in X$, $d(x, y) + d(y, z) \geq d(x, z)$. (X, d) is said to be a finite metric space if X is finite. (X, d) is called a semi-metric space if one allows $d(x, y) = 0$ even when $x \neq y$.*

Definition A.2 (X_1, d_1) *embeds with distortion at-most* Γ *into* (X_2, d_2) *if there exists a map* $\phi : X_1 \mapsto X_2$ *such that for all* $x, y \in X$ $d_1(x,y) \leq d_2(\phi(x), \phi(y)) \leq \Gamma \cdot d_1(x,y)$. *If* $\Gamma = 1$, *then* (X_1, d_1) *is said to* **isometrically** *embed in* (X_2, d_2).

The metrics we would be concerned with are:
(1) ℓ_p *metrics*: For $X \subseteq \mathbb{R}^m$, for some $m \geq 1$, and $\mathbf{x}, \mathbf{y} \in X$, $\ell_p(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^m |x_i - y_i|^p\right)^{1/p}$. Here, $p \geq 1$, and the metric $\ell_\infty(\mathbf{x}, \mathbf{y}) = \max_{i=1}^m |x_i - y_i|$.
(2) *Cut (semi-)metrics*: A cut metric δ_S on a set X, defined by the set $S \subseteq X$ is:

$$\delta_S(x,y) = \begin{cases} 1 & \text{if } |\{\mathrm{x}, \mathrm{y}\} \cap \mathrm{S}| = 1 \\ 0 & \text{otherwise} \end{cases}$$

The cut-cone (denoted CUT_n) is the cone generated by cut metrics on an n-point set X. Formally, $\mathsf{CUT}_n := \left\{\sum_S \lambda_S \delta_S : \lambda_S \geq 0 \text{ for all } \mathrm{S} \subseteq \mathrm{X}\right\}$. To avoid referring to the dimension, denote $\mathsf{CUT} := \cup_n \mathsf{CUT}_n$.
(3) *Negative type metrics*: A metric space (X, d) is said to be of negative type if $(X, \sqrt{d})$ embeds isometrically into ℓ_2. Formally, there is an integer m and a vector $\mathbf{v}_x \in \mathbb{R}^m$ for every $x \in X$, such that $d(x,y) = \|\mathbf{v}_x - \mathbf{v}_y\|^2$. The class of all negative type metrics is denoted by ℓ_2^2.

A.2. Facts about Metric Spaces

Fact A.3 *[11] Any finite metric space isometrically embeds into* ℓ_∞.

Fact A.4 *[11]* (X, d) *is* ℓ_1 *embeddable iff* $d \in \mathsf{CUT}$.

Fact A.5 *[11]* $\ell_1 \subseteq \ell_2^2$.

Theorem A.6 (Bourgain's Embedding Theorem [7]) *Any* n*-point metric space embeds into* ℓ_1 *with distortion at-most* $C_{\mathrm{b}} \log n$, *for some absolute constant* C_{b}.

Fact A.7 *[6, 25] There is an* n*-point metric, any embedding of which into* ℓ_1, *requires* $\Omega(\log n)$ *distortion.*

A.3. The $(\ell_2^2, \ell_1, O(1))$-Conjecture

Conjecture A.8 ($(\ell_2^2, \ell_1, O(1))$-Conjecture, [16, 24]) *Every* ℓ_2^2 *metric can be embedded into* ℓ_1 *with distortion at-most* C_{neg}, *for some absolute constant* $C_{\mathrm{neg}} \geq 1$.

A.4. Cut Problems and their SDP Relaxations

In this section, we define the cut problems that we study and present their SDP relaxations. All graphs are complete undirected graphs and may have multiple edges or self loops, unless mentioned otherwise, with non-negative *weights* or *demands* associated to its edges. For a graph $G = (V, E)$, and $S \subseteq V$, let $E(S, \overline{S})$ denote the set of edges with one endpoint in S and other in $\overline{S}$.

Remark A.9 *The versions of* SPARSEST CUT *and* BALANCED SEPARATOR *that we define below are non-uniform versions with demands. The uniform version has all demands equal to* 1 *(i.e. unit demand for every pair of vertices).*

The Sparsest Cut Problem

Definition A.10 (SPARSEST CUT) *For a graph* $G = (V, E)$ *with a weight* $\mathbf{wt}(e)$, *and a demand* $\mathbf{dem}(e)$ *associated to each edge* $e \in E$, *the goal is to optimize* $\min_{\emptyset \neq S \subsetneq V} \frac{\sum_{e \in E(S, \overline{S})} \mathbf{wt}(e)}{\sum_{e \in E(S,\overline{S})} \mathbf{dem}(e)}$.

It follows from Fact A.4 that the objective function above is the same as $\min_{d \in \ell_1} \frac{\sum_{e=\{x,y\} \in E} \mathbf{wt}(e) d(x,y)}{\sum_{e=\{x,y\} \in E} \mathbf{dem}(e) d(x,y)}$. Denote this minimum for $\{G, \mathbf{wt}, \mathbf{dem}\}$ by $\psi_1(G)$. Consider the following two quantities associated to $\{G, \mathbf{wt}, \mathbf{dem}\}$:

$$\psi_\infty(G) := \min_{d \in \ell_\infty} \frac{\sum_{e=\{x,y\} \in E} \mathbf{wt}(e) d(x,y)}{\sum_{e=\{x,y\} \in E} \mathbf{dem}(e) d(x,y)},$$

$$\psi_{\mathrm{neg}}(G) := \min_{d \in \ell_2^2} \frac{\sum_{e=\{x,y\} \in E} \mathbf{wt}(e) d(x,y)}{\sum_{e=\{x,y\} \in E} \mathbf{dem}(e) d(x,y)}.$$

Facts A.5 and A.3 imply that $\psi_1(G) \geq \psi_{\mathrm{neg}}(G) \geq \psi_\infty(G)$. In addition, Bourgain's Embedding Theorem (Theorem A.6) can be used to show that $\psi_1(G) \leq O(\log n) \cdot \psi_\infty(G)$, where $n := |V|$. Fact A.7 implies that this factor of $O(\log n)$ is tight upto a constant.

It is also the case that $\psi_{\mathrm{neg}}(G)$ is efficiently computable using a semi-definite program (SDP) of Figure 2. Let the vertex set be $V = \{1, 2, \ldots, n\}$. For a metric d on V, let $\mathbf{Q} := \mathbf{Q}(d)$ be the matrix whose (i,j)-th entry is $\mathbf{Q}[i,j] := \frac{1}{2}\left(d(i,n) + d(j,n) - d(i,j)\right)$.

$$\text{Minimize} \quad \sum_{e=\{i,j\}} \mathbf{wt}(e) d(i,j)$$

$$\begin{array}{lrl} \text{Subject to} & d(\cdot,\cdot) & \text{is a metric} \\ & \multicolumn{2}{c}{\sum_{e=\{i,j\}} \mathbf{dem}(e) d(i,j) = 1} \\ & \mathbf{Q}(d) & \text{is positive semidefinite} \end{array}$$

Figure 2. SDP relaxation of SPARSEST CUT

Fact A.11 *Suppose that every* n*-point* ℓ_2^2 *metric embeds into* ℓ_1 *with distortion* $f(n)$. *Then,* $\psi_1(G) \leq f(n) \cdot \psi_{neg}(G)$, *and* SPARSEST CUT *can be approximated to within a factor of* $f(n)$. *In particular, if the* $(\ell_2^2, \ell_1, O(1))$*-Conjecture (Conjecture A.8) is true, then there is a constant factor approximation algorithm for* SPARSEST CUT.

The Balanced Separator Problem

Definition A.12 (BALANCED SEPARATOR) *For a graph $G = (V, E)$ with a weight $\mathbf{wt}(e)$, and a demand $\mathbf{dem}(e)$ associated to each edge $e \in E$, let $D := \sum_{e \in E} \mathbf{dem}(e)$ be the total demand. Let a* **balance** *parameter B be given where $D/4 \leq B \leq D/2$. The goal is to find a non-trivial cut $(S, \overline{S})$ that minimizes $\sum_{e \in E(S,\overline{S})} \mathbf{wt}(e)$, subject to $\sum_{e \in E(S,\overline{S})} \mathbf{dem}(e) \geq B$. The cuts that satisfy $\sum_{e \in E(S,\overline{S})} \mathbf{dem}(e) \geq B$ are called B-***balanced cuts**.

Figure 3 is an SDP relaxation of BALANCED SEPARATOR with parameter B.

$$\text{Minimize} \quad \frac{1}{4} \sum_{e=\{x,y\}} \mathbf{wt}(e) \|\mathbf{v}_x - \mathbf{v}_y\|^2$$

Subject to

$$\begin{aligned} \forall x \in V \qquad & \|\mathbf{v}_x\|^2 = 1 \\ \forall\, x, y, z \in V \qquad & \|\mathbf{v}_x - \mathbf{v}_y\|^2 + \|\mathbf{v}_y - \mathbf{v}_z\|^2 \geq \|\mathbf{v}_x - \mathbf{v}_z\|^2 \\ & \tfrac{1}{4} \textstyle\sum_{e=\{x,y\}} \mathbf{dem}(e) \|\mathbf{v}_x - \mathbf{v}_y\|^2 \geq B \end{aligned}$$

Figure 3. SDP **relaxation of** BALANCED SEPARATOR

We need the following (folk-lore) result stating that, starting with a good SDP solution to Figure 3, one can find a balanced partition in a graph by iteratively finding (approximate) sparsest cut in the graph.

Theorem A.13 *Suppose $x \mapsto \mathbf{v}_x$ is a solution for the* SDP *of Figure 3 with objective value $\frac{1}{4} \sum_{e=\{x,y\}} \mathbf{wt}(e) \|\mathbf{v}_x - \mathbf{v}_y\|^2 \leq \varepsilon$. Assume that the ℓ_2^2 metric defined by the vectors $\{\mathbf{v}_x |\ x \in V\}$ embeds into ℓ_1 with distortion $f(n)$ ($n = |V|$). Then, there exists a B'-balanced cut $(S, \overline{S})$, $B' \geq B/3$ such that $\sum_{e \in E(S,\overline{S})} \mathbf{wt}(e) \leq O(f(n) \cdot \varepsilon)$.*

The ARV-Conjecture

Conjecture A.14 (Uniform Version) *The integrality gap of the* SDP *of Figure 1 is $O(1)$.*

Conjecture A.15 (Non-Uniform Version) *The integrality gap of the* SDP *of Figure 3 is $O(1)$.*

Fact A.16 *The $(\ell_2^2, \ell_1, O(1))$-Conjecture implies the non-uniform* ARV*-Conjecture (Conjecture A.15).*

The Maximum Cut Problem

Definition A.17 (MAXIMUM CUT) *For a graph $G = (V, E)$ with a weight $\mathbf{wt}(e)$ associated to each edge $e \in E$, the goal is to optimize $\max_{\emptyset \neq S \subsetneq V} \frac{\sum_{e \in E(S,\overline{S})} \mathbf{wt}(e)}{\sum_{e \in E} \mathbf{wt}(e)}$.*

$$\text{Maximize} \quad \frac{1}{4} \sum_{e=\{x,y\}} \mathbf{wt}(e) \|\mathbf{v}_x - \mathbf{v}_y\|^2$$

Subject to

$$\begin{aligned} \forall x \in V \qquad & \|\mathbf{v}_x\|^2 = 1 \\ \forall\, x, y, z \in V \qquad & \|\mathbf{v}_x - \mathbf{v}_y\|^2 + \|\mathbf{v}_y - \mathbf{v}_z\|^2 \geq \|\mathbf{v}_x - \mathbf{v}_z\|^2 \end{aligned}$$

Figure 4. SDP **relaxation of** MAXIMUM CUT

Goemans and Williamson [17] gave an α_{GW} (≈ 0.878) approximation algorithm for MAXIMUM CUT. They showed that every SDP solution with objective value γ_{SDP} can be rounded to a cut in the graph that cuts edges with weight $\geq \alpha_{\mathsf{GW}} \cdot \gamma_{\mathsf{SDP}}$. We note here that their rounding procedure does not make use of the triangle inequality constraints.

The Minimum Uncut Problem

Definition A.18 (MINIMUM UNCUT) *Given a graph $G = (V, E)$ with a weight $\mathbf{wt}(e)$ associated to each edge $e \in E$, the goal is to optimize $\min_{\emptyset \neq S \subsetneq V} \frac{\sum_{e \in E(S,S) \cup E(\overline{S},\overline{S})} \mathbf{wt}(e)}{\sum_{e \in E} \mathbf{wt}(e)}$.*

The semi-definite relaxation of MINIMUM UNCUT is similar to that of MAXIMUM CUT except the objective function in Figure 4 is replaced by

$$\text{Minimize} \quad \left(1 - \quad \frac{1}{4} \sum_{e=\{x,y\}} \mathbf{wt}(e) \|\mathbf{v}_x - \mathbf{v}_y\|^2\right)$$

Goemans and Williamson [17] showed that every SDP solution for MINIMUM UNCUT with objective value β_{SDP} can be rounded to a cut in the graph, such that the weight of edges left uncut is at-most $O(\sqrt{\beta_{\mathsf{SDP}}})$. We note again that their rounding procedure does not make use of the triangle inequality constraints.

B. Overall Strategy for Disproving the $(\ell_2^2, \ell_1, O(1))$-Conjecture

We describe the high-level approach to our disproval of the $(\ell_2^2, \ell_1, O(1))$-Conjecture in this section. We construct an integrality gap instance of non-uniform BALANCED SEPARATOR to disprove the non-uniform ARV-Conjecture, and that suffices to disprove the $(\ell_2^2, \ell_1, O(1))$-Conjecture using the (folk-lore) Fact A.16.

We construct a complete weighted graph $G(V, \mathbf{wt})$, with vertex set V and weight $\mathbf{wt}(e)$ on edge e, and with $\sum_e \mathbf{wt}(e) = 1$. The vertex set is partitioned into sets $V_1, V_2, \ldots, V_r$, each of size $|V|/r$ (think of $r \approx \sqrt{|V|}$).

A cut A in the graph is viewed as a function $A : V \mapsto \{-1, 1\}$. We are interested in cuts that cut *many* sets V_i in a *somewhat balanced* way: For $0 \le \theta \le 1$, a cut A is called θ-piecewise balanced if $\mathbf{E}_{i \in_R [r]}\left[\,\left|\mathbf{E}_{x \in_R V_i}[A(x)]\right|\,\right] \le \theta$. We also assign a unit vector to every vertex in the graph. Let $\mathbf{v}_x$ denote the vector assigned to vertex x. Our construction of the graph $G(V, \mathbf{wt})$ and the vector assignment $x \mapsto \mathbf{v}_x$ can be summarized as follows:

Theorem B.1 *Fix any $\frac{1}{2} < t < 1$. For every sufficiently small $\varepsilon > 0$, there exists a graph $G(V, \mathbf{wt})$, with a partition $V = \cup_{i=1}^r V_i$, and a vector assignment $x \mapsto \mathbf{v}_x$ for every $x \in V$, such that: (1) $|V| \le 2^{2^{O(1/\varepsilon^3)}}$, (2) Every $\frac{5}{6}$-piecewise balanced cut A must cut ε^t fraction of edges, i.e., for any such cut $\sum_{e \in E(A,\overline{A})} \mathbf{wt}(e) \ge \varepsilon^t$, (3) The unit vectors $\{\mathbf{v}_x \mid x \in V\}$ define a negative type metric, i.e., the following triangle inequality is satisfied: $\forall\, x, y, z \in V,\ \|\mathbf{v}_x - \mathbf{v}_y\|^2 + \|\mathbf{v}_y - \mathbf{v}_z\|^2 \ge \|\mathbf{v}_x - \mathbf{v}_z\|^2$, (4) For each part V_i, the vectors $\{\mathbf{v}_x \mid x \in V_i\}$ are* **well-separated***, i.e., $\frac{1}{2}\mathbf{E}_{x,y \in_R V_i}\left[\|\mathbf{v}_x - \mathbf{v}_y\|^2\right] = 1$ (4) The vector assignment gives a* **low** SDP *objective value, i.e., $\frac{1}{4}\sum_{e=\{x,y\}} \mathbf{wt}(e)\|\mathbf{v}_x - \mathbf{v}_y\|^2 \le \varepsilon$.*

Theorem B.2 *The $(\ell_2^2, \ell_1, O(1))$-Conjecture is false. In fact, for every $\delta > 0$, for all sufficiently large n, there are n-point ℓ_2^2 metrics that require distortion at-least $(\log\log n)^{1/6-\delta}$ to embed into ℓ_1.*

Proof: Suppose that the ℓ_2^2 metric defined by vectors $\{\mathbf{v}_x | \, x \in V\}$ in Theorem B.1 embeds into ℓ_1 with distortion Γ. We will show that $\Gamma = \Omega\left(\frac{1}{\varepsilon^{1-t}}\right)$ using Theorem A.13.

Construct an instance of BALANCED SEPARATOR as follows. The graph $G(V, \mathbf{wt})$ is as in Theorem B.1. The demands $\mathbf{dem}(e)$ depend on the partition $V = \cup_{i=1}^r V_i$. We let $\mathbf{dem}(e) = 1$ if e has both endpoints in the same part V_i for some $1 \le i \le r$, and $\mathbf{dem}(e) = 0$ otherwise. Clearly, $D := \sum_e \mathbf{dem}(e) = r \cdot \binom{|V|/r}{2}$.

Now, $x \mapsto \mathbf{v}_x$ is an assignment of unit vectors that satisfy the triangle inequality constraints. This will be a solution to the SDP of Figure 3. Property (4) of Theorem B.1 guarantees that $\frac{1}{4}\sum_{e=\{x,y\}} \mathbf{dem}(e)\|\mathbf{v}_x - \mathbf{v}_y\|^2 = \frac{1}{4} \cdot r \cdot \binom{|V|/r}{2} \cdot 2 = D/2 =: B$. Thus, the SDP solution is $D/2$-balanced and its objective value is at-most ε. Using Theorem A.13, we get a B'-balanced cut $(A, \overline{A})$, $B' \ge D/6$ such that $\sum_{e \in E(A,\overline{A})} \mathbf{wt}(e) \le O(\Gamma \cdot \varepsilon)$. Using the Cauchy-Schwartz Inequality, it is easy to see that the cut $(A, \overline{A})$ must be a $\frac{5}{6}$-piecewise balanced cut.

However, Property (2) of Theorem B.1 says that such a cut must cut at-least ε^t fraction of edges. This implies that $\Gamma = \Omega(\frac{1}{\varepsilon^{1-t}})$. The theorem follows by noting that $t > \frac{1}{2}$ is arbitrary and $n := |V| \le 2^{2^{O(1/\varepsilon^3)}}$.

C. The Unique Games Conjecture (UGC)

Definition C.1 (UNIQUE GAMES) *An instance $\mathcal{U}\left(G(V,E), [N], \{\pi_e\}_{e \in E}, \mathbf{wt}\right)$ of* UNIQUE GAMES *is defined as follows: $G = (V, E)$ is a a graph with a set of vertices V and a set of edges E, with possibly parallel edges. An edge e whose endpoints are v and w is written as $e\{v, w\}$. For every $e \in E$, there is a bijection $\pi_e : [N] \mapsto [N]$, and a weight $\mathbf{wt}(e) \in \mathbb{R}^+$. For an edge $e\{v, w\}$, we think of π_e as a pair of permutations $\{\pi_e^v, \pi_e^w\}$, where $\pi_e^w = (\pi_e^v)^{-1}$. π_e^v is a mapping that takes a label of vertex w to a label of vertex v. The goal is to assign one* label *to every vertex of the graph from the set $[N]$. The labeling is supposed to satisfy the constraints given by bijective maps π_e. A labeling $\lambda : V \mapsto [N]$* satisfies *an edge $e\{v, w\}$, if $\lambda(v) = \pi_e^v(\lambda(w))$. Define the indicator function $I^\lambda(e)$, which is 1 if e is satisfied by λ and 0 otherwise. The optimum $\mathsf{OPT}(\mathcal{U})$ of the* UNIQUE GAMES *instance is defined to be $\max_\lambda \sum_{e \in E} \mathbf{wt}(e) \cdot I^\lambda(e)$. Without loss of generality, we assume that $\sum_{e \in E} \mathbf{wt}(e) = 1$.*

Conjecture C.2 (UGC **[19]**) *For every pair of constants $\eta, \zeta > 0$, there exists a sufficiently large constant $N := N(\eta, \zeta)$, such that it is* NP*-hard to decide whether a* UNIQUE GAMES *instance $\mathcal{U}\left(G(V,E), [N], \{\pi_e\}_{e \in E}, \mathbf{wt}\right)$, has $\mathsf{OPT}(\mathcal{U}) \ge 1 - \eta$, or $\mathsf{OPT}(\mathcal{U}) \le \zeta$.*

Consider a UNIQUE GAMES instance $\mathcal{U} = \left(G(V,E), [N], \{\pi_e\}_{e \in E}, \mathbf{wt}\right)$. Khot [19] proposed the SDP in Figure 5. This SDP was inspired by a paper of Feige and Lovasz [14]. Here, for every $u \in V$, we associate a set of N orthogonal vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_N\}$. The intention is that if $i_0 \in [N]$ is a label for vertex $u \in V$, then $\mathbf{u}_{i_0} = \sqrt{N}\mathbf{1}$, and $\mathbf{u}_i = \mathbf{0}$ for all $i \neq i_0$. Here, $\mathbf{1}$ is some fixed unit vector and $\mathbf{0}$ is the zero-vector. However, once we take the SDP relaxation, this may no longer be true and $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N\}$ could be any set of orthogonal vectors.

$$\text{Maximize} \quad \sum_{e\{u,v\} \in E} \mathbf{wt}(e) \cdot \frac{1}{N}\left(\sum_{i=1}^N \langle \mathbf{u}_{\pi_e^u(i)}, \mathbf{v}_i \rangle\right)$$

Subject to

$$\begin{array}{rr}
\forall\, u \in V & \langle \mathbf{u}_1, \mathbf{u}_1\rangle + \cdots + \langle \mathbf{u}_N, \mathbf{u}_N\rangle = N \\
\forall\, u \in V \ \ \forall\ i \neq j & \langle \mathbf{u}_i, \mathbf{u}_j\rangle = 0 \\
\forall\, u, v \in V \ \ \forall\ i, j & \langle \mathbf{u}_i, \mathbf{v}_j\rangle \ge 0 \\
\forall\, u, v \in V & \sum_{1 \le i,j \le N} \langle \mathbf{u}_i, \mathbf{v}_j\rangle = N
\end{array}$$

Figure 5. SDP **relaxation of** UNIQUE GAMES