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| **Spectral Graph Theory and its Applications** | **out:** September 9, 2004 |

# Problem Set 1

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| *Lecturer: Daniel A. Spielman* | **due:** September 21, 2004 |

I have tried to arrange these problems in order of difficulty. They can all be solved using material covered in the lectures so far.

1a. Prove that if $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of $L_G$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ are the eigenvalues of $L_H$, then $\lambda_i \neq \mu_i$ implies that $G$ and $H$ are non-isomorphic.

1b. Prove that $G$ and $H$ are isomorphic if and only if there exists a permutation matrix $P$ such that $A_H = P A_G P^T$.

1c. Let $\lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of $A_G$. Assume that $\lambda_i$ is isolated, that is $\lambda_{i-1} < \lambda_i < \lambda_{i+1}$ and let $v_i$ be the corresponding eigenvector. Let $H$ be a graph isomorphic to $G$, and let $w_i$ be the $i$th eigenvector of $G$. Then, there exists a $c \in \{1, -1\}$ and a permutation $\pi$ such that $v_i(j) = c w_i(\pi(j))$.

2. For $d$ a positive integer and $n = 2^d$, let $G_n$ be the graph with vertex set $\{0,1\}^d$ in which each pair of vertices that differ in at most two coordinates are joined by an edge. Prove upper and lower bounds on $\lambda_2(G_n)$. Make them as close to each other as possible.

3. For the complete binary tree $T_n$, prove that $\lambda_2(T_n) \geq 1/cn$ for some absolute constant $c$. (Hint: use the full power of Lemma 3.2.2)

4. Let $c_1, \ldots, c_{n-1} > 0$ and let $P$ be the weighted path graph with Laplacian

$$L_P = \sum_{i=1}^{n-1} c_i L_{(i,i+1)}.$$

You will show that a test vector can be used to prove a lower bound on $\lambda_2$! That is, let $v$ be any vector such that

$$v(1) < v(2) < \cdots < v(n).$$

Prove that

$$\lambda_2(P) \geq \min_{i : v_i \neq 0} \frac{(L_P v)_i}{v_i}.$$