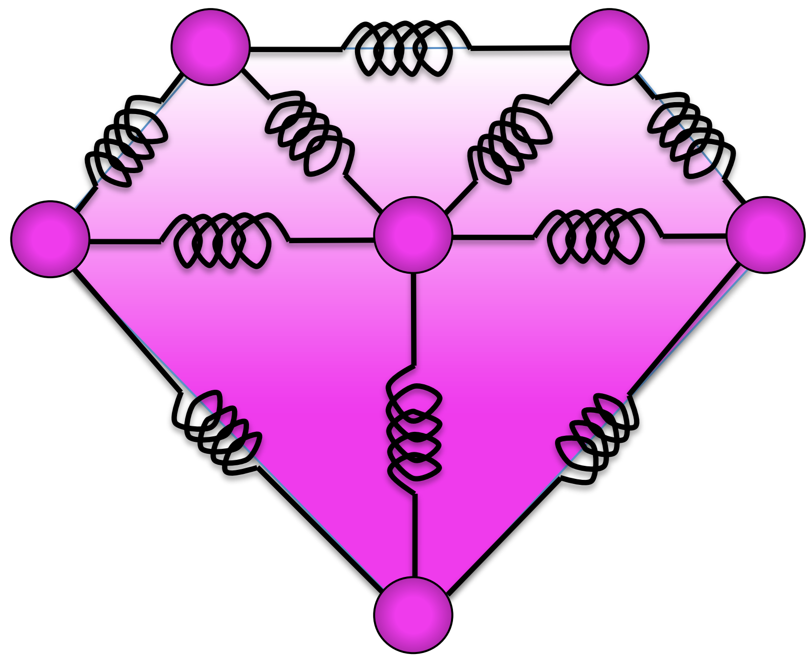
Laplacian Gems



Daniel A. Spielman
Yale University

Outline

Laplacian Linear Systems (Electrical Graph Theory)

Applications

Fast Algorithms

- Bold ideas
- Surprising results
- Powerful techniques

Faster than Koutis-Miller-Peng?

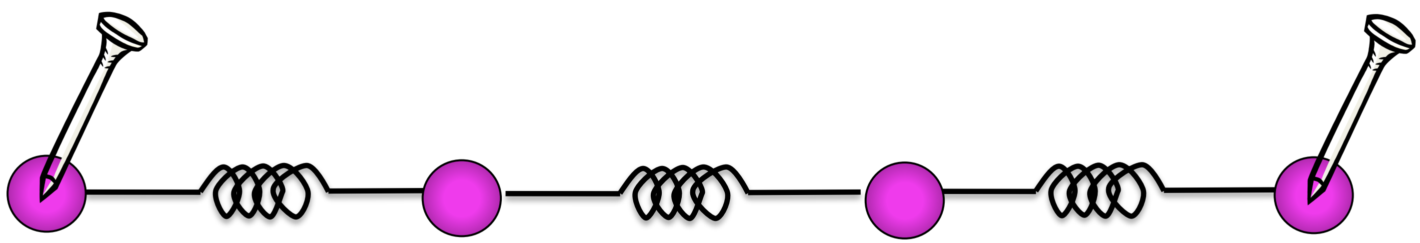
Graphs as Spring Networks

edges -> ideal linear springs
weights -> spring constants (k)

Physics: when stretched to length x, force is kx

potential energy is $kx^2/2$

Nail down some vertices, let rest settle



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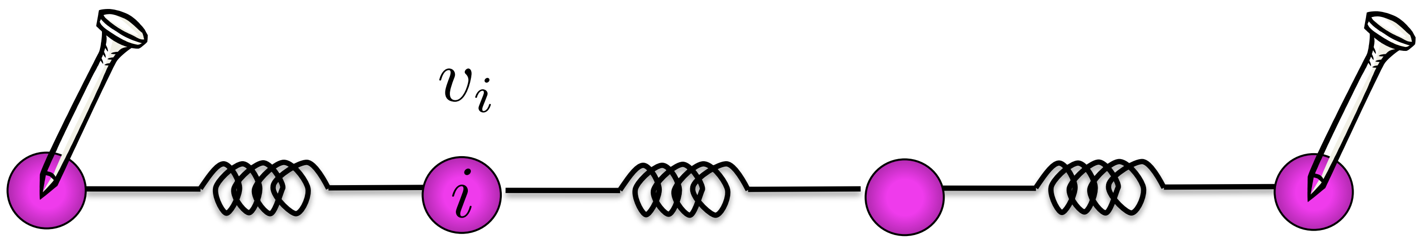


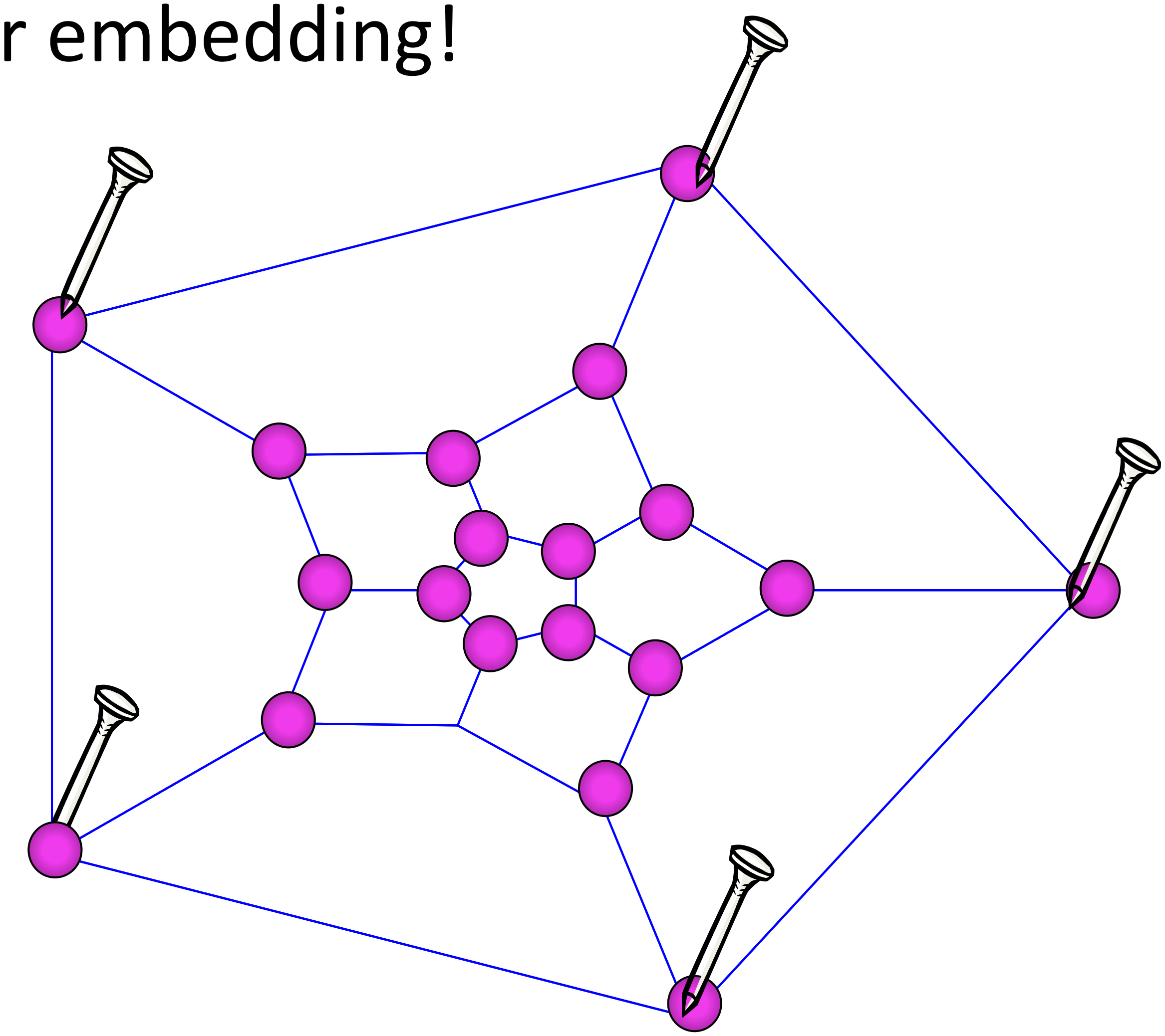
Physics: minimizes total potential energy

$$\sum_{(i,j)\in E} w_{i,j}(v_i - v_j)^2 = v^T L v$$

subject to boundary constraints (nails)

Tutte's Theorem '63

If nail down a face of a planar 3-connected graph, get a planar embedding!



The Laplacian

$$v^T L v = \sum_{(i,j)\in E} w_{i,j}(v_i - v_j)^2$$

$$L_{i,j} = \begin{cases} -w_{i,j} & \text{if } (i,j) \in E \\ d_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad d_i = \sum_j w_{i,j}$$

$$L = \sum_{(i,j)\in E} w_{i,j}(e_i - e_j)(e_i - e_j)^T$$

The Laplacian

$$v^T L v = \sum_{(i,j) \in E} w_{i,j} (v_i - v_j)^2$$

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Can minimize, subject to boundary $v_i = f_i \text{ for } i \in S$

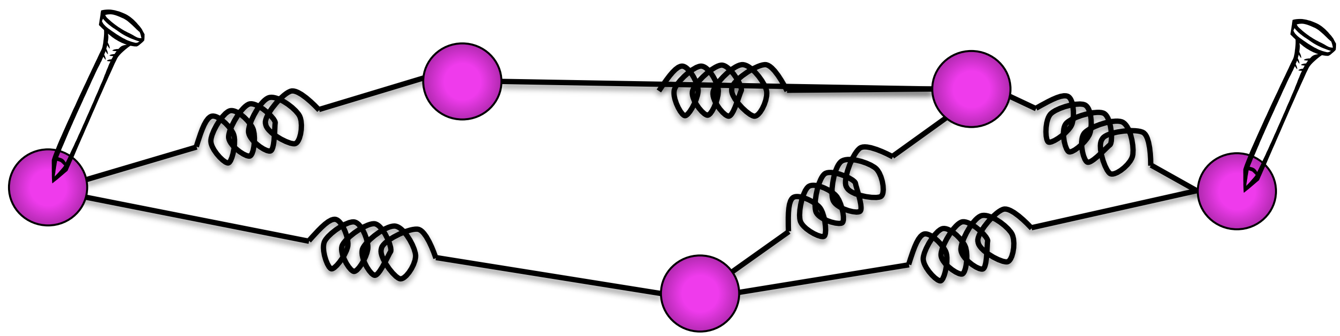
by Laplacian Linear Solvers in nearly-linear time

[Koutis-Miller-Peng '10, S-Teng '04]

Regression on Graphs [Zhu-Ghahramani-Lafferty '03]

Say know f(i) for all i in S $\subset$ V
want to guess f(j) for all other j.

Idea: nail node i to f(i) on Real line.



$$\min \sum_{(i,j)\in E} w_{i,j}(v_i - v_j)^2 = v^T L v$$

Graphs as Resistor Networks

edge -> resistor
weight -> 1/resistance

Ohm's law: flow/current

$$f_{i,j} = \frac{v_i - v_j}{r_{i,j}} = w_{i,j}(v_i - v_j)$$

Graphs as Resistor Networks

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Demands/external current:

$$d = Lv \qquad\qquad v = L^+ d$$

the pseudo-inverse

Graphs as Resistor Networks

Ohm's law: flow/current

$$f_{i,j} = \frac{v_i - v_j}{r_{i,j}} = w_{i,j}(v_i - v_j)$$

Demands/external current:

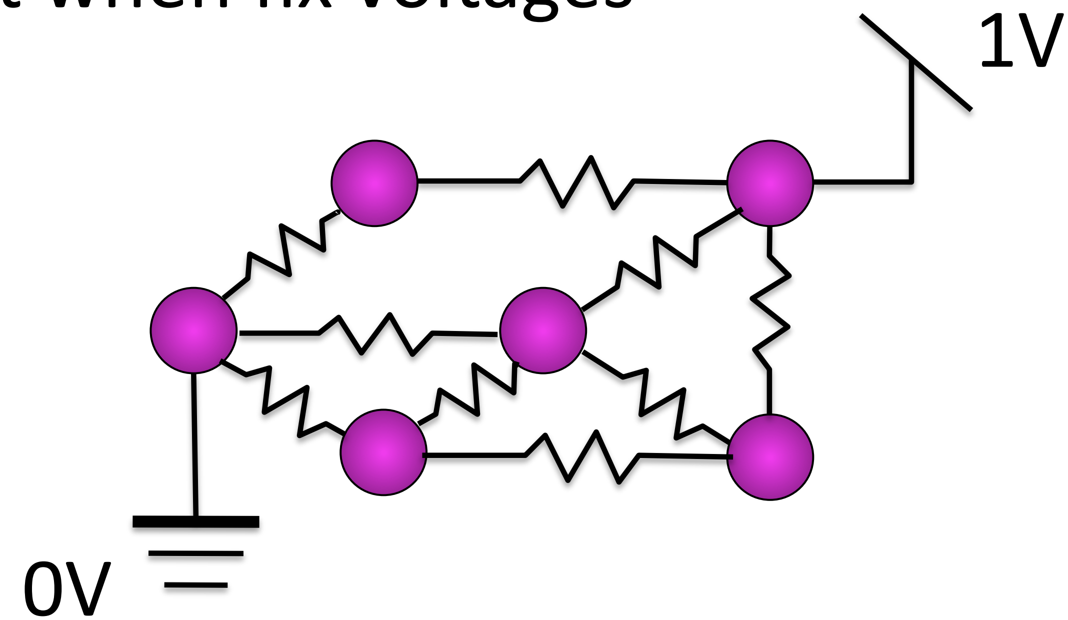
$$d = Lv \qquad\qquad v = L^+ d$$

To flow 1 from s to t:

- Set d(s) = 1 and d(t) = -1
- Solve for v
- Use Ohm's law to find flow in graph

Graphs as Resistor Networks

To solve for current when fix voltages



Solve for v minimizing dissipated energy,

$$v^T L v = \sum w_{i,j}(v_i - v_j)^2$$

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Graphs as Resistor Networks

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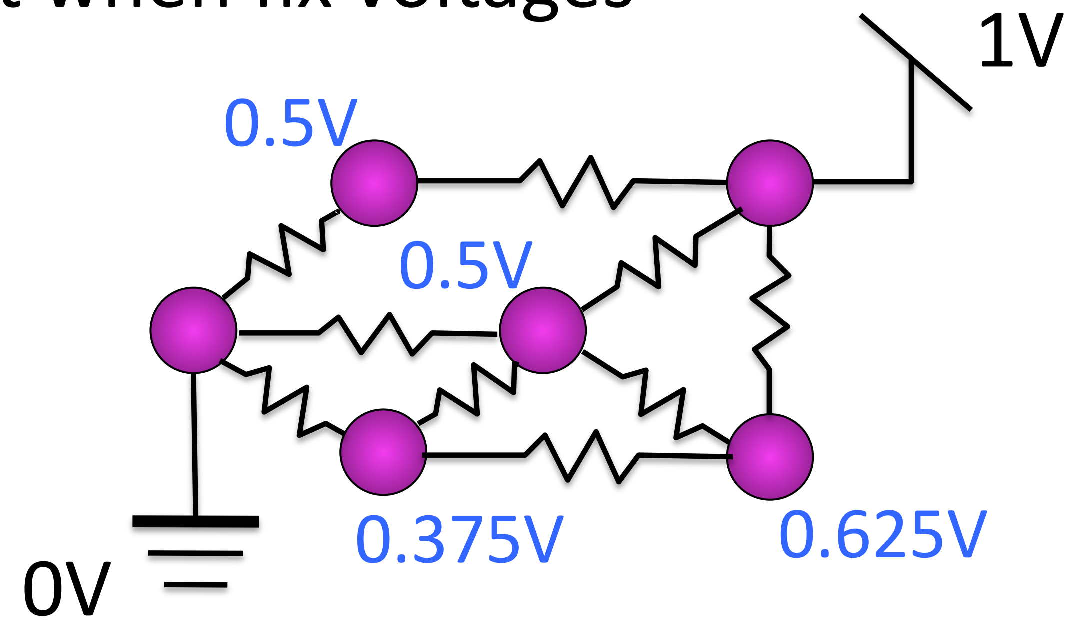

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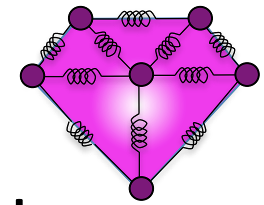
Effective Resistance between s and t



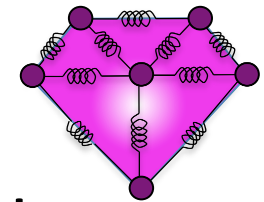
= resistance of whole network between s and t

= $v_s - v_t$ in s-t flow of value 1

$$= (e_s - e_t)^T L^+ (e_s - e_t)$$

$$f_{i,j} = \frac{v_i - v_j}{r_{i,j}}$$

Effective Resistance between s and t



= resistance of whole network between s and t

= $v_s - v_t$ in s-t flow of value 1

$$= (e_s - e_t)^T L^+ (e_s - e_t)$$

= 1/effective spring constant between s and t

= prob edge (s,t) in a random spanning tree

~ expected commute time between s and t
[Chandra-Raghavan-Ruzzo-Smolensky-Tiwari '89]

And, is a distance.

Application: Random Spanning Trees

Can use fast Laplacian solvers to sample random spanning trees in time $\tilde{O}(mn^{1/2})$ [Kelner-Madry-Propp '09]

Idea: accelerate random-walk based sampling [Aldous, Broder] by jumping through clusters where walk is slow.

Application: Approximate Maximum Flow

[Christiano-Kelner-Madry-S-Teng '10]

1. Compute electrical s-t flow
2. Increase resistance on over-capacity edges
3. Return average of computed flows (a la MWM)

Time $\tilde{O}(m^{3/2}/\epsilon^3)$

Application: Approximate Maximum Flow

[Christiano-Kelner-Madry-S-Teng '10]

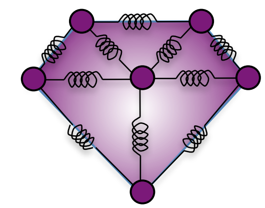
1. Compute electrical s-t flow
2. Increase resistance on over-capacity edges
   and delete very over-capacity edges
3. Return average of computed flows (a la MWM)

Time $\tilde{O}(m^{3/2}/\epsilon^3) \longrightarrow \tilde{O}(mn^{1/3}/\epsilon^{11/3})$

Approximate Laplacian Solvers

Preconditioned Conjugate Gradient [Hestenes '51, Stiefel '52, ???]	$O(mn)$
Vaidya '90: Subgraph preconditioners	$O(mn^{3/4})$
Boman-Hendrickson '01: Using Low-Stretch Spanning Trees	$\tilde{O}(mn^{1/2})$
S-Teng '04: Spectral sparsification	$\tilde{O}(m)$
Koutis-Miller-Peng '10: Elegance	$O(m \log^2 n)$

Preconditioned Conjugate Gradient



Solve $L_G x = b$ by

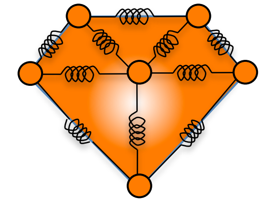
Approximating L_G by L_H (the preconditioner)

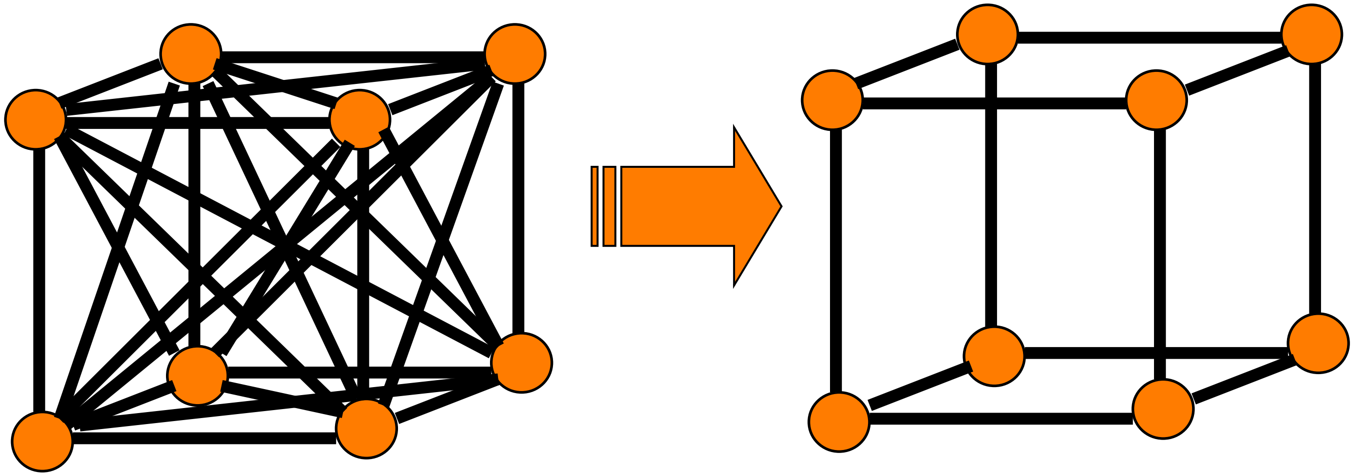
In each iteration
solve a system in L_H
multiply a vector by L_G

ϵ-approx solution after
$O(\sqrt{\kappa(L_G, L_H)} \log \epsilon^{-1})$ iterations

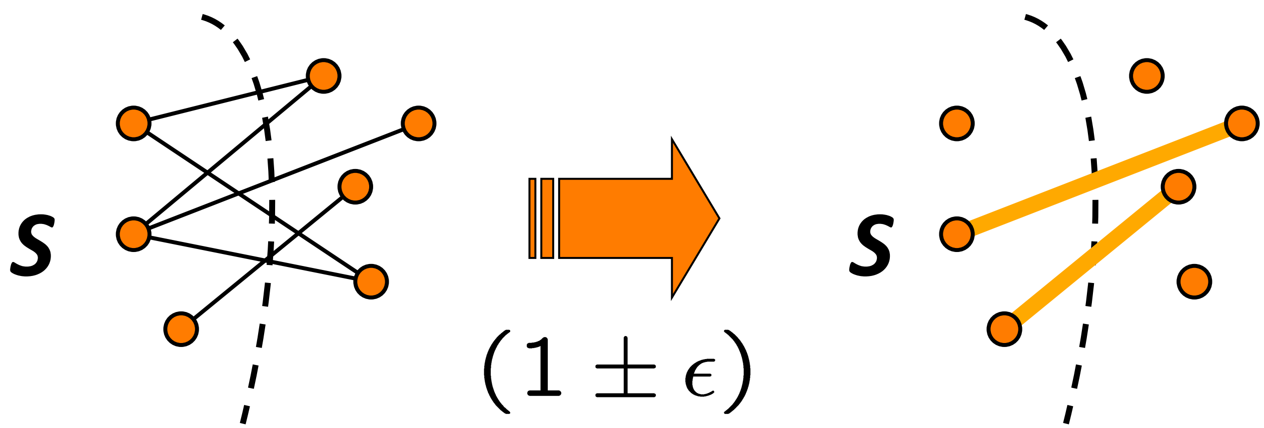
condition number/approx quality

Graph Sparsification [Benczur-Karger '96]

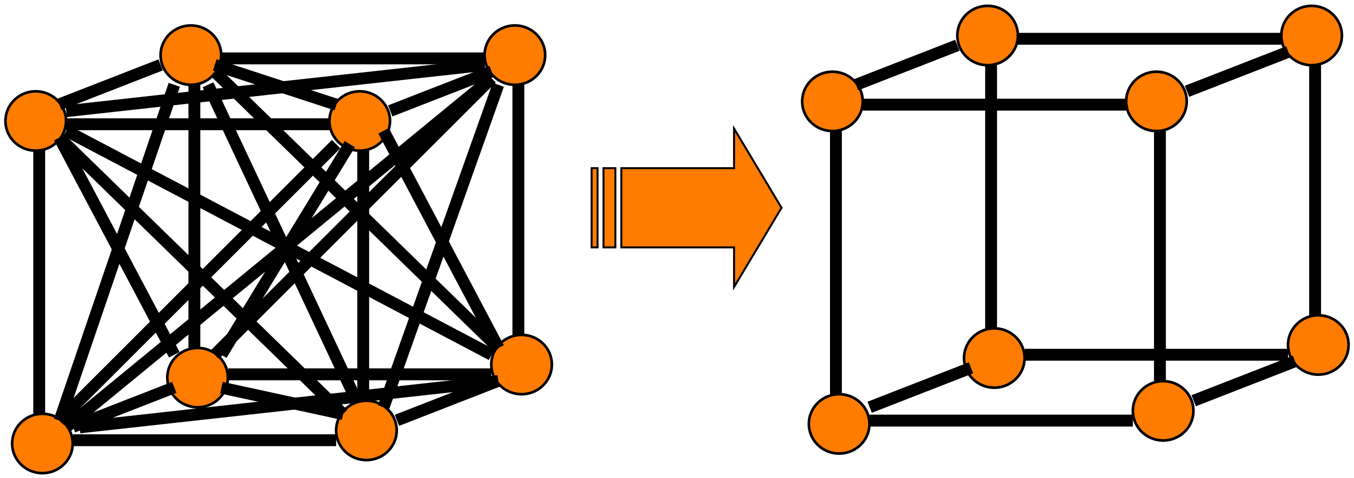




Approximate G by a sparse H,
approximately preserving all cuts



Spectral Sparsification [S-Teng '04]



Approximate G by a sparse H,
approximately preserving all energies
and effective resistances

$$v^T L_G v \quad \xrightarrow[(1 \pm \epsilon)]{} \quad v^T L_H v$$

Spectral Sparsification [S-Srivastava '08]

$$v^T L_G v \quad \underset{(1 \pm \epsilon)}{\Longrightarrow} \quad v^T L_H v$$

Sample edges (i,j) of G
with probability proportional to $R_{eff}(i,j) w_{i,j}$

$O(n \log n / \epsilon^2)$ edges suffice

Analysis by random matrix concentration results
of Rudelson and Vershynin.

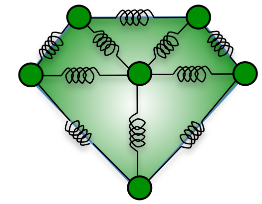
Spectral Sparsification [Koutis-Miller-Peng '10]

Understand Rudelson-Vershynin much better

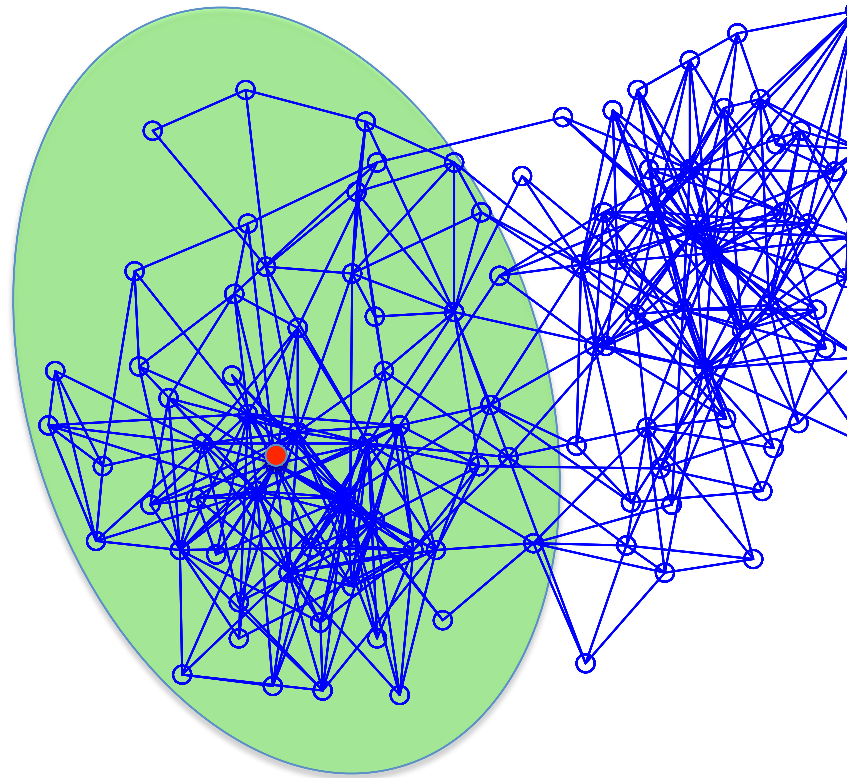
Suffices to sample edges (i,j) of G
with probability crudely related to $R_{eff}(i,j)w_{i,j}$

Sample $O(n \log n/\epsilon^2)$ edges ***with replacement***
so actually get many fewer edges!

Local Graph Clustering [S-Teng '04]



Given vertex of interest find nearby cluster S with small conductance in time O(|S|)



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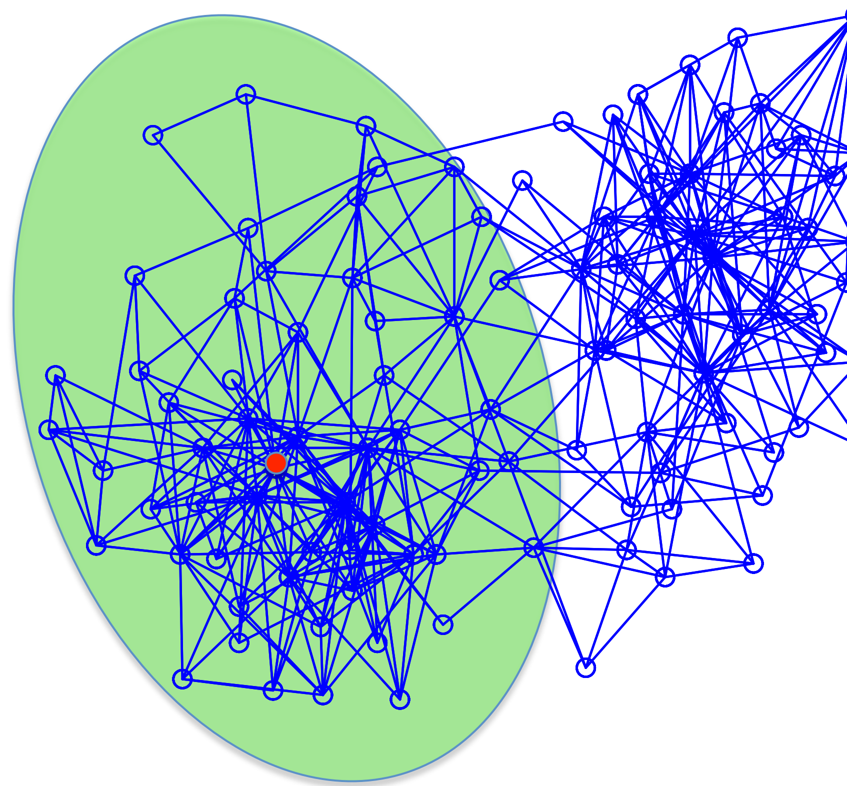

How should one explore a graph?

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Possible goals:

find nodes like the first node

find a representative sample

minimize computation and graph access

prove something rigorous

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Beat BFS

Local Graph Clustering [S-Teng '04]

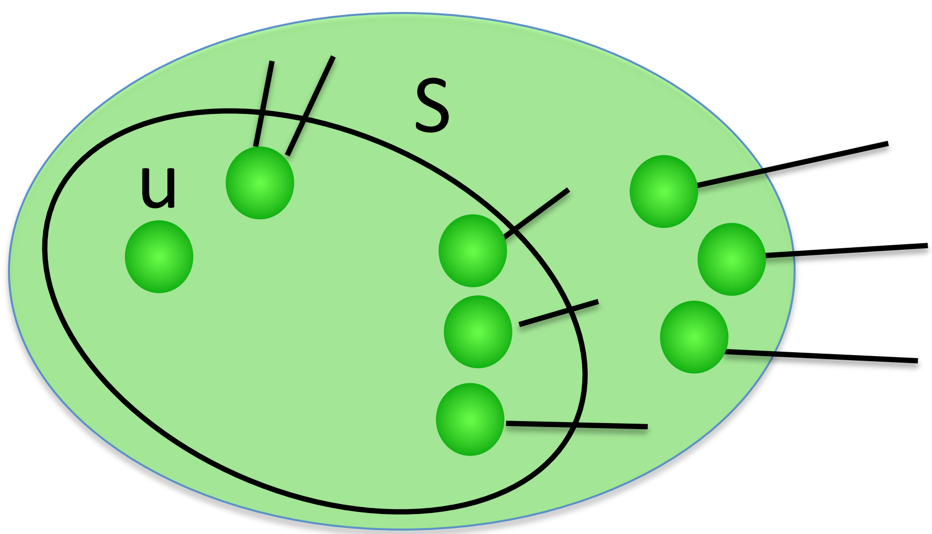
Prove: if S has small conductance ϕ

u is a random node in S

probably

find a set of small conductance, $\phi^{1/2} \log^c n$

in time $|S| \log^c n / \phi^2$



Using Approximate Personal PageRank Vectors

Jeh-Widom '03, Berkhin '06, Andersen-Chung-Lang '06

Spilling paint in a graph:

start at one node

at each step,

α fraction dries

of wet paint, half stays put, half to neighbors

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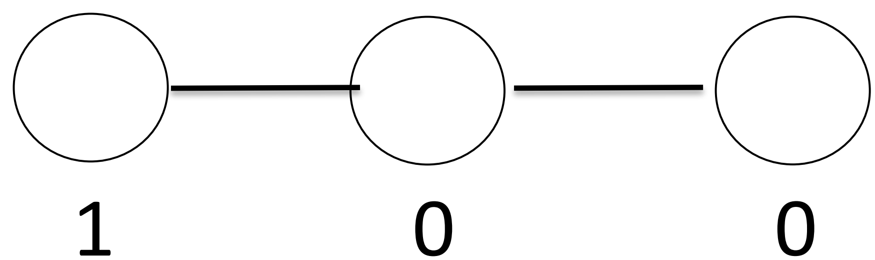

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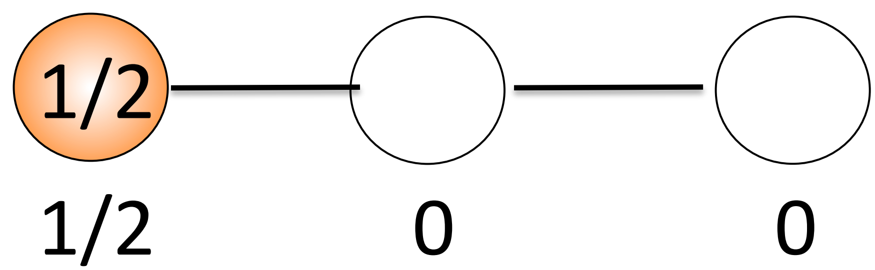

with $\alpha = 1/2$

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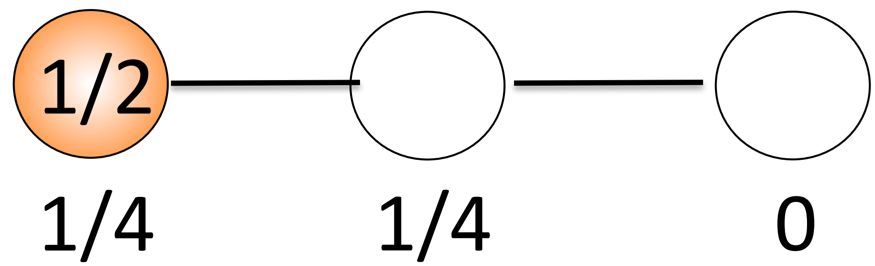

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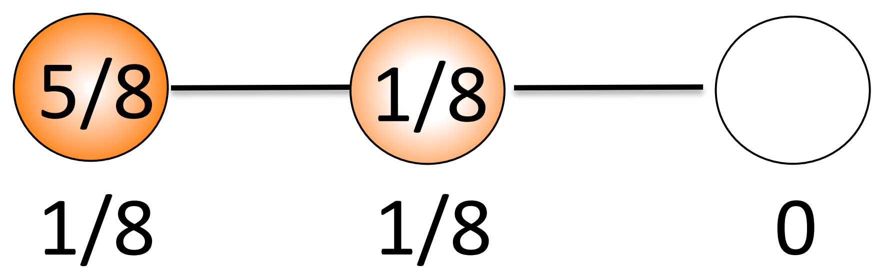

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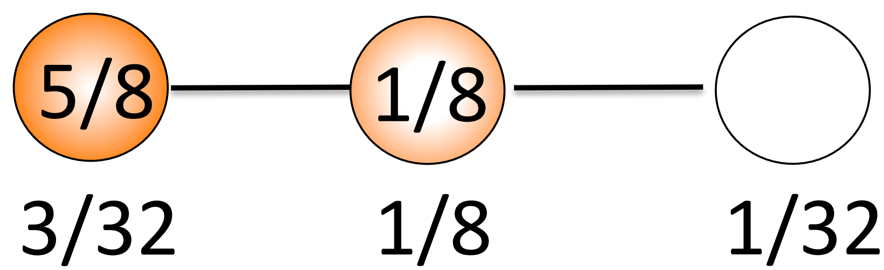

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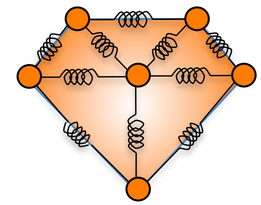

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Time doesn’t matter, can push paint whenever

Approximate: only push when a lot of paint

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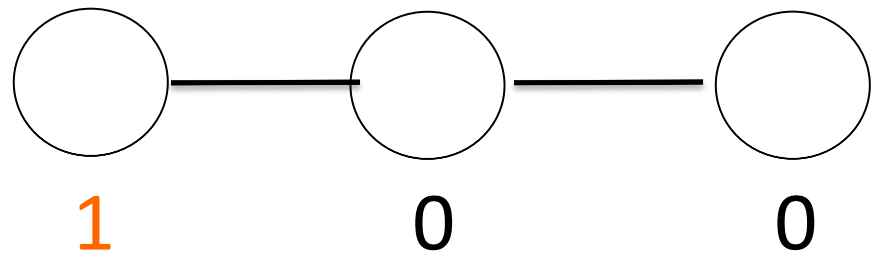

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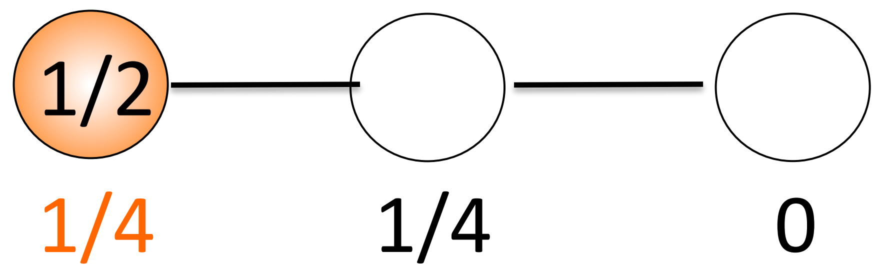

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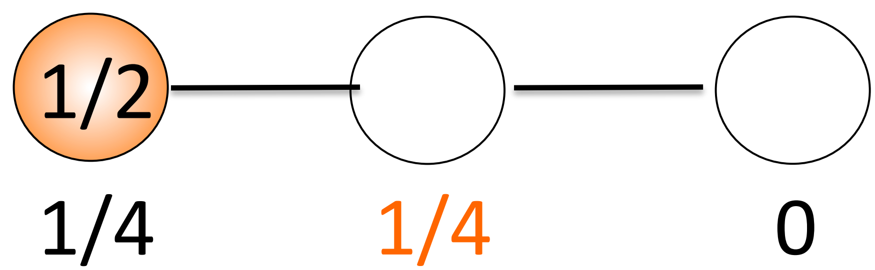

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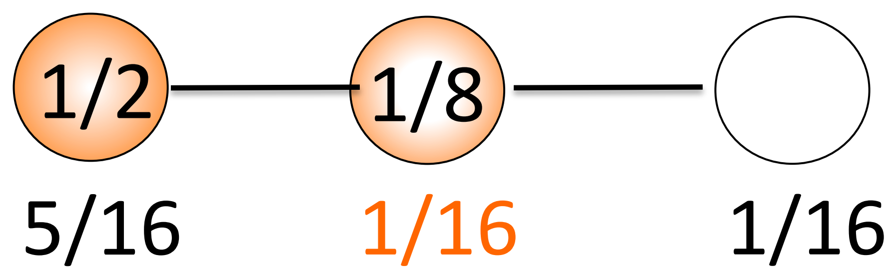

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Volume-Biased Evolving Set Markov Chain

[Andersen-Peres '09]

Walk on sets of vertices
starts at one vertex, ends at V

Dual to random walk on graph

When start inside set of conductance ϕ
find set of conductance $\phi^{1/2} \log^{1/2} n$
with work $|S| \log^c n / \phi^{1/2}$

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When start inside set of conductance ϕ
 find set of conductance $\phi^{1/2} \log^{1/2} n$
 with work $|S| \log^c n / \underline{\phi^{1/2}}$

can we eliminate this?

A Technique for proving convergence of walks on graphs of Lovász and Simonovits

Consider lazy random walk on a regular graph (stays put with probability ½).

Approaches uniform in time 1/conductance^2.

A Technique for proving convergence of walks on graphs of Lovász and Simonovits

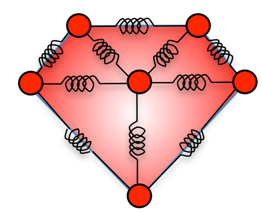
Consider lazy random walk on a regular graph (stays put with probability ½).

Approaches uniform in time 1/conductance^2.

$p^t(i)$ = prob of vertex i at time t

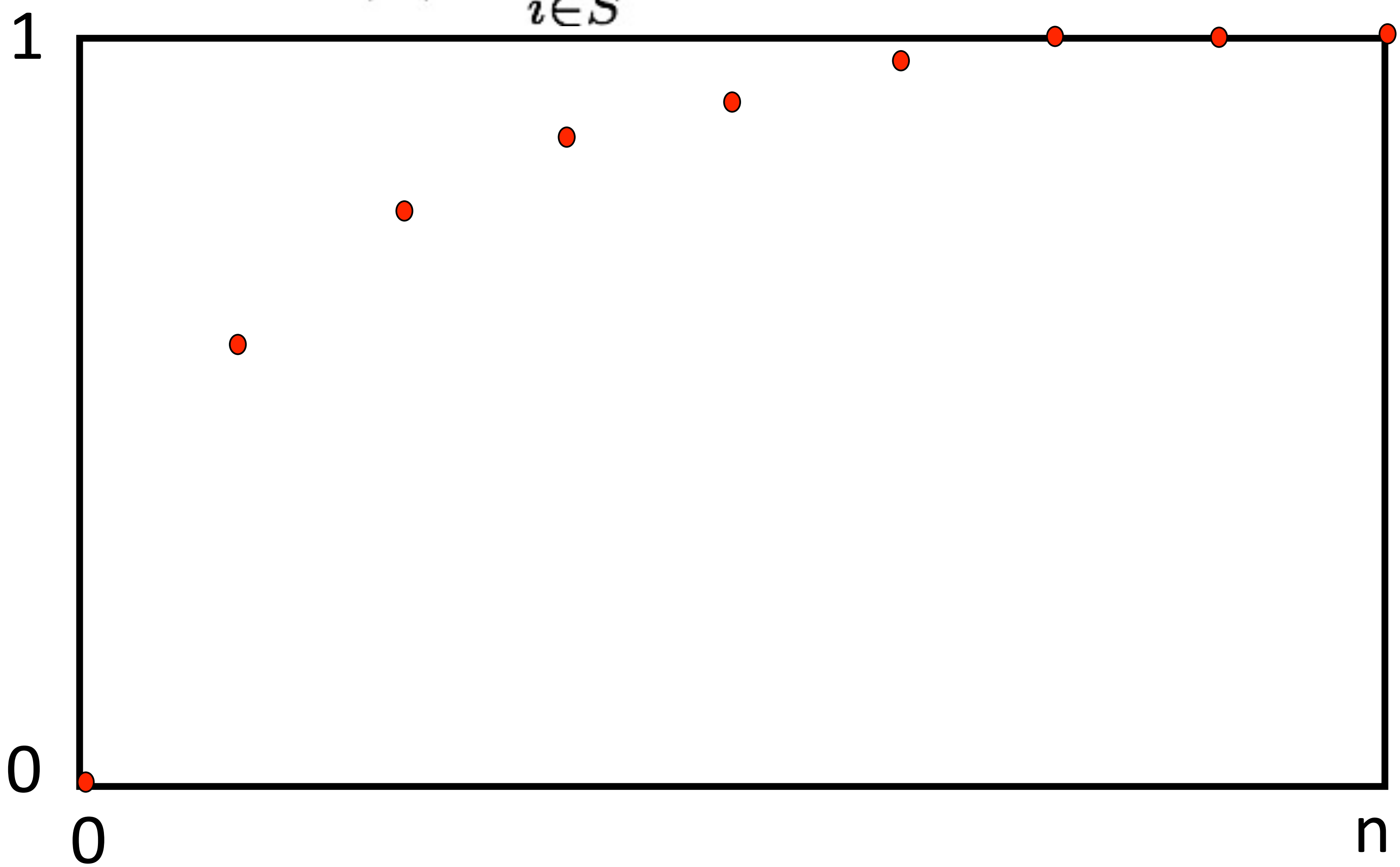
$$C^t(k) = \max_{|S|=k} \sum_{i \in S} p^t(i)$$

sum of largest k values



Lovász-Simonovits

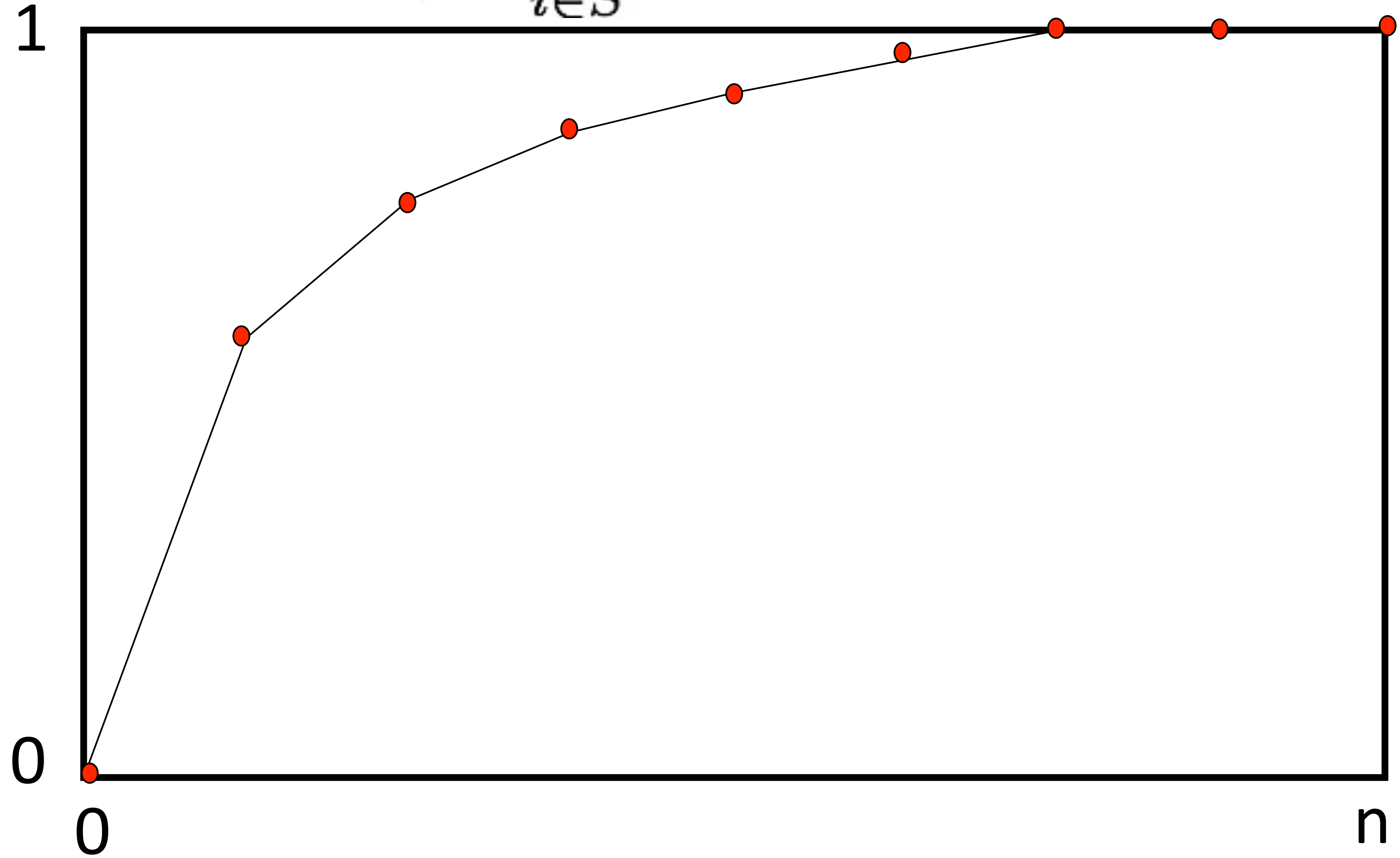
$$C^t(k) = \max_{|S|=k} \sum_{i \in S} p^t(i)$$



Lovász-Simonovits

$$C^t(k) = \max_{|S|=k} \sum_{i \in S} p^t(i)$$

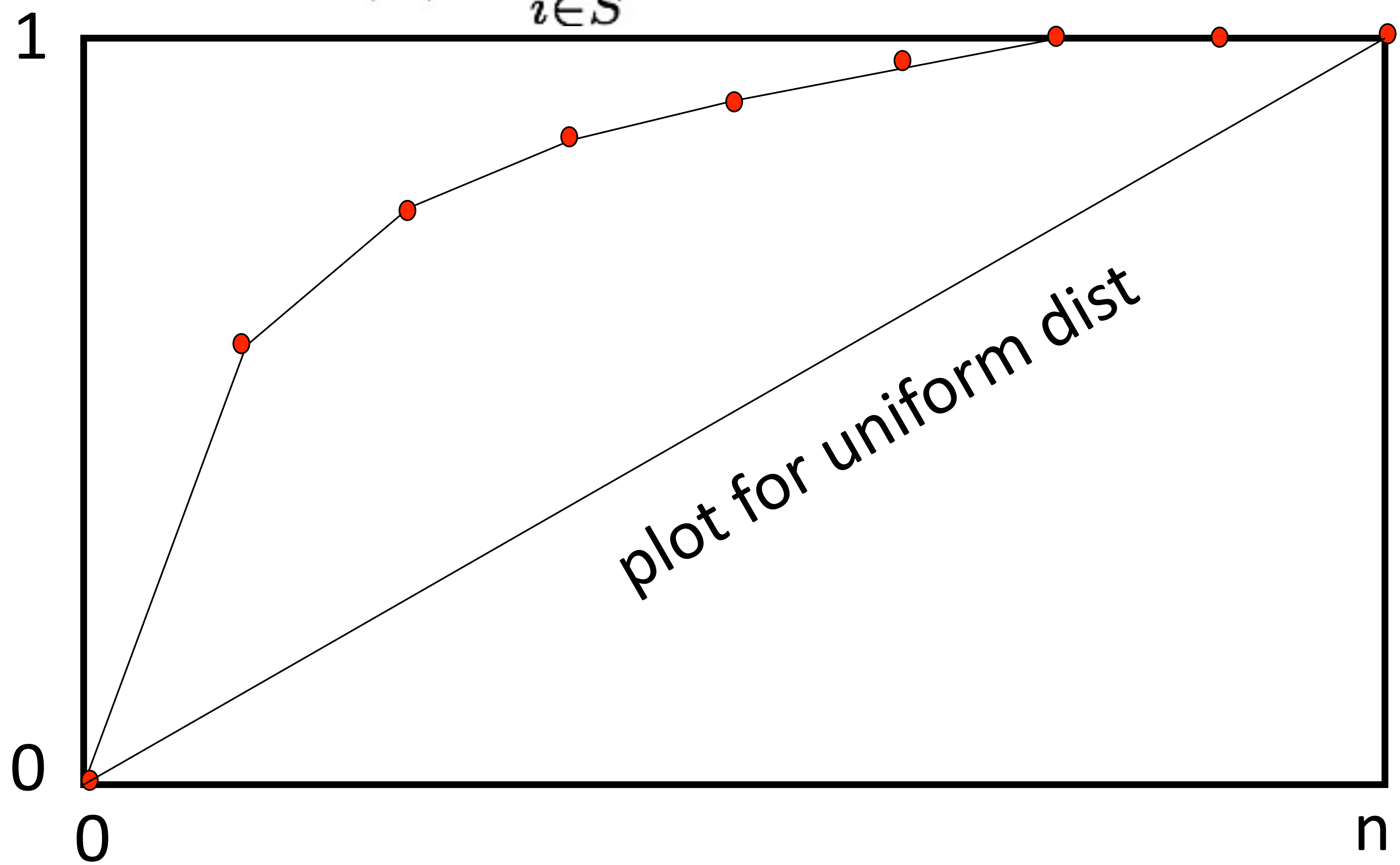
extend by linearity



Lovász-Simonovits

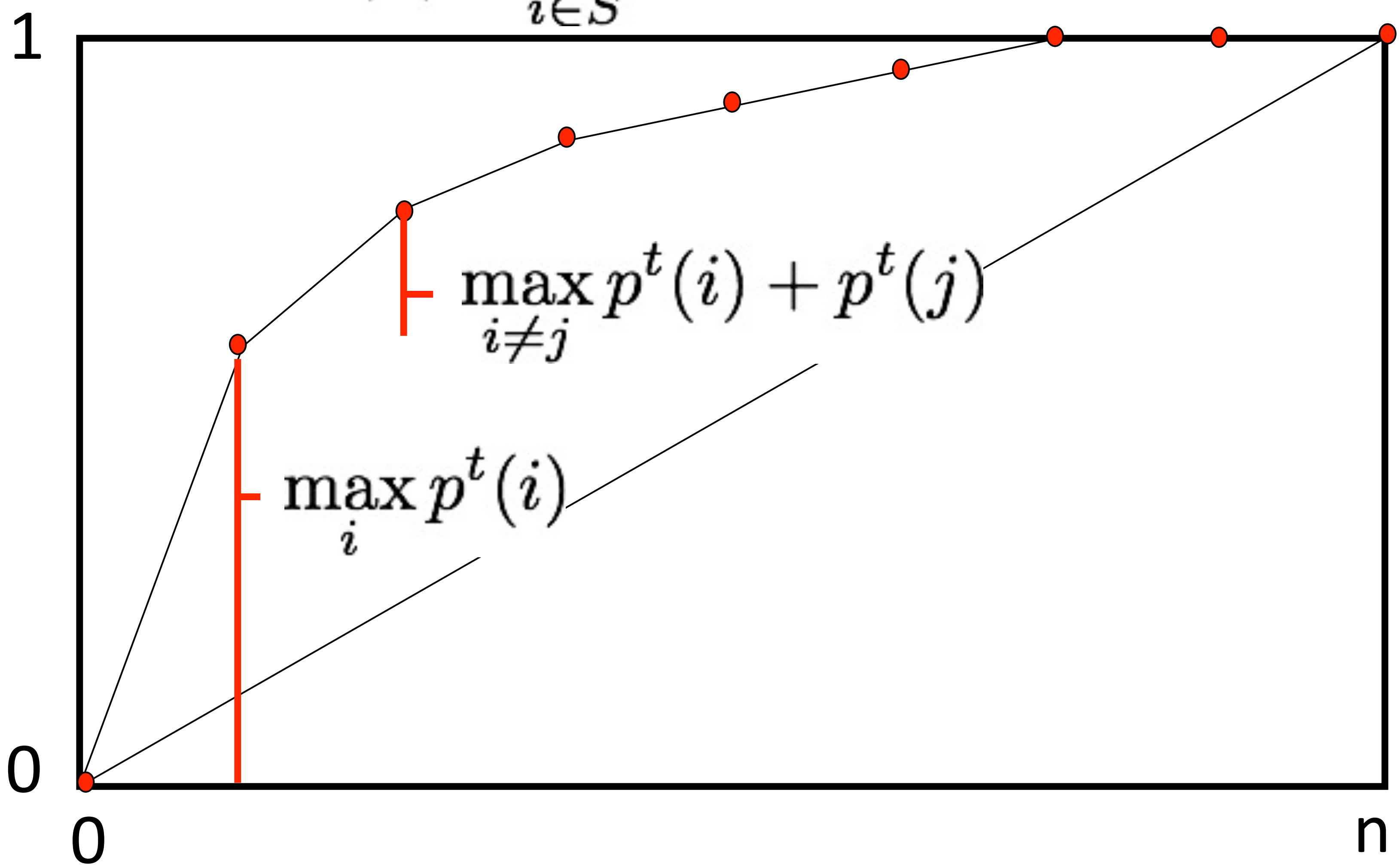
$$C^t(k) = \max_{|S|=k} \sum_{i \in S} p^t(i)$$

interpolate linearly



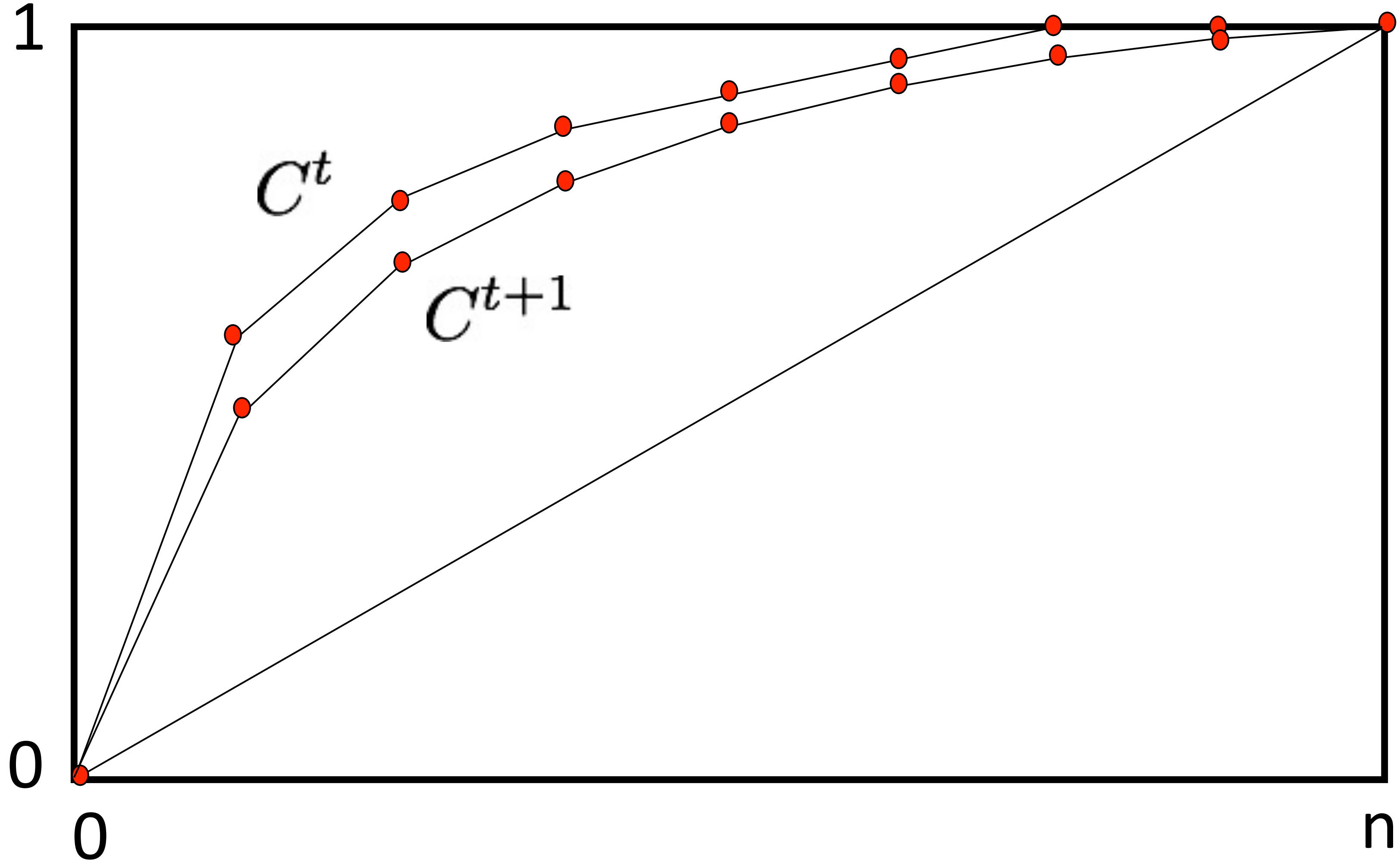
Lovász-Simonovits

$$C^t(k) = \max_{|S|=k} \sum_{i \in S} p^t(i)$$ is concave



Lovász-Simonovits: Easy inequality

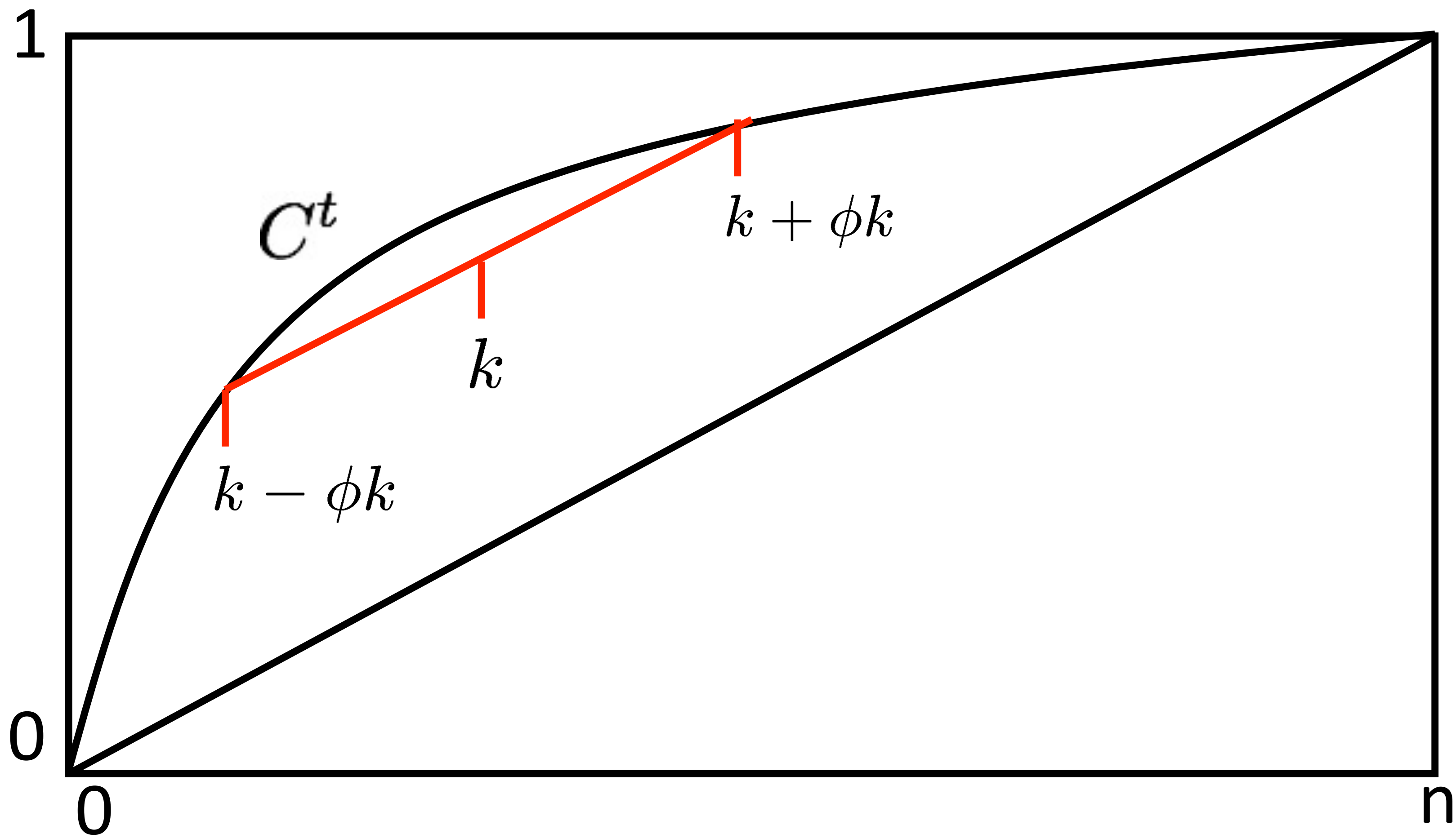
$$C^{t+1} \leq C^t$$



Lovász-Simonovits: Conductance ϕ inequality

$$C^{t+1}(k) \le \frac{1}{2}\left(C^t(k-\phi k) + C^t(k+\phi k)\right) \qquad k \in [0, n/2]$$

$$C^{t+1}(k) \le \frac{1}{2}\left(C^t(k-\phi(n-k)) + C^t(k+\phi(n-k))\right) \quad k \in [n/2, n]$$



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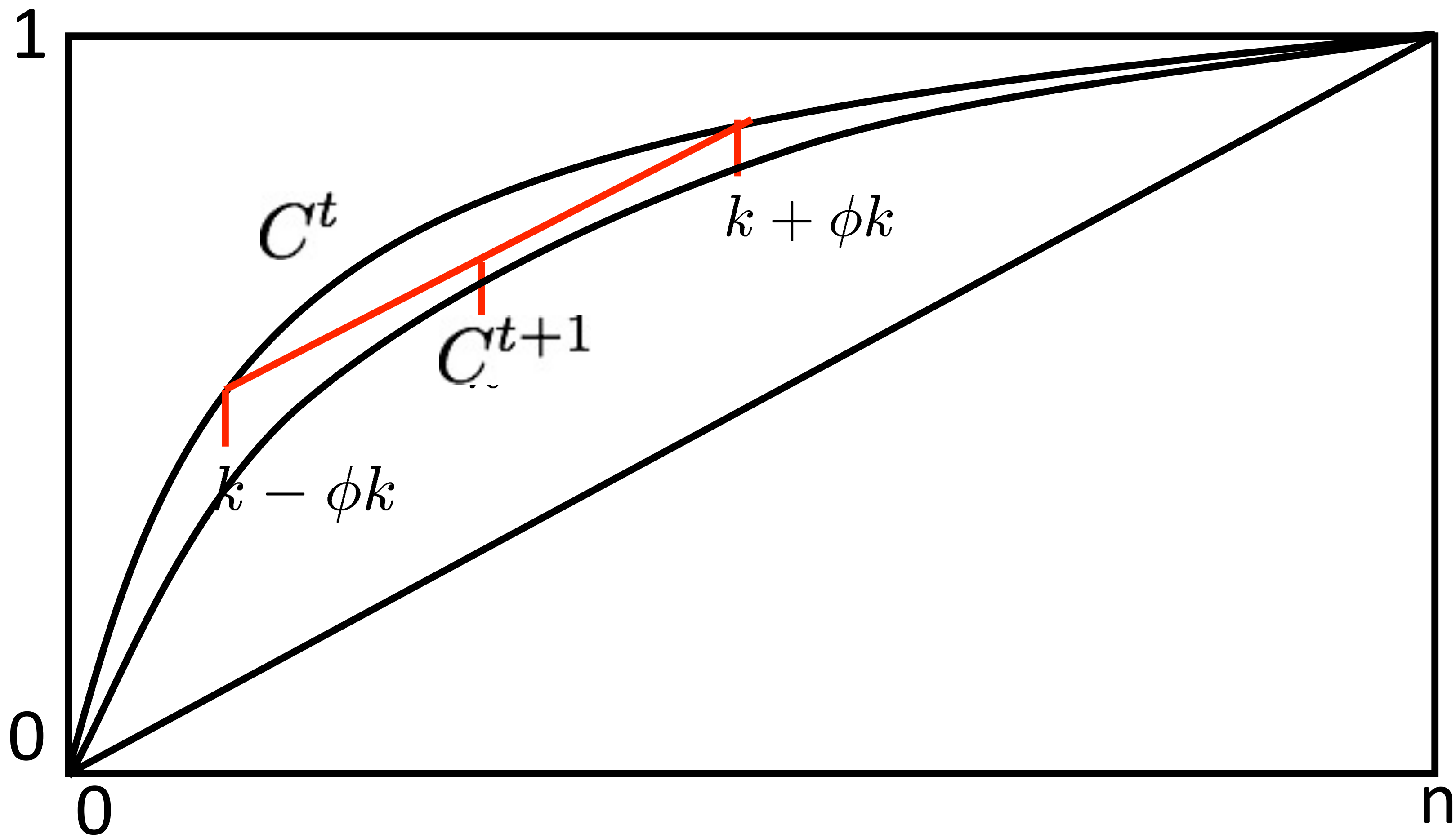

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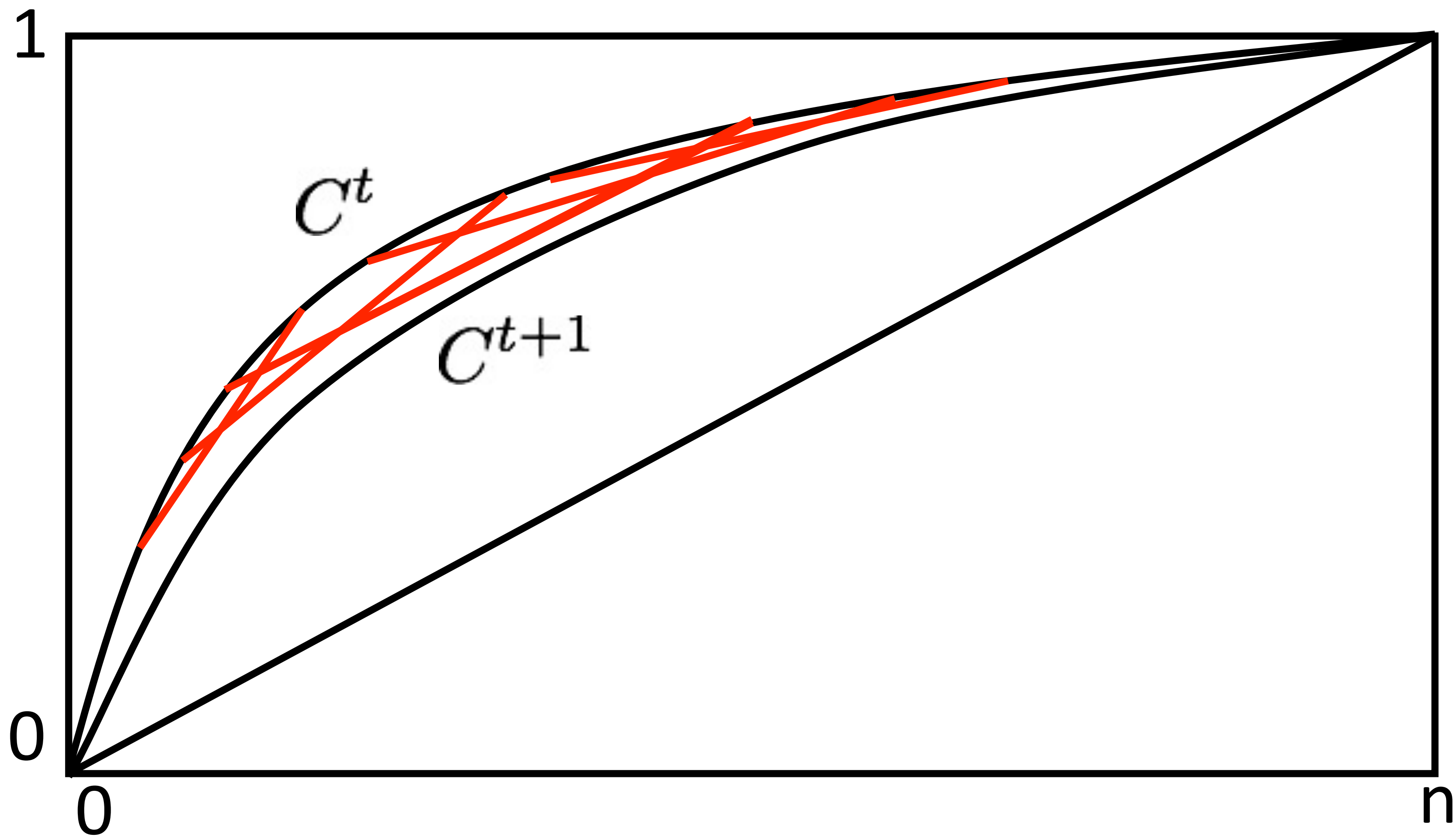

Lovász-Simonovits: Conductance φ inequality

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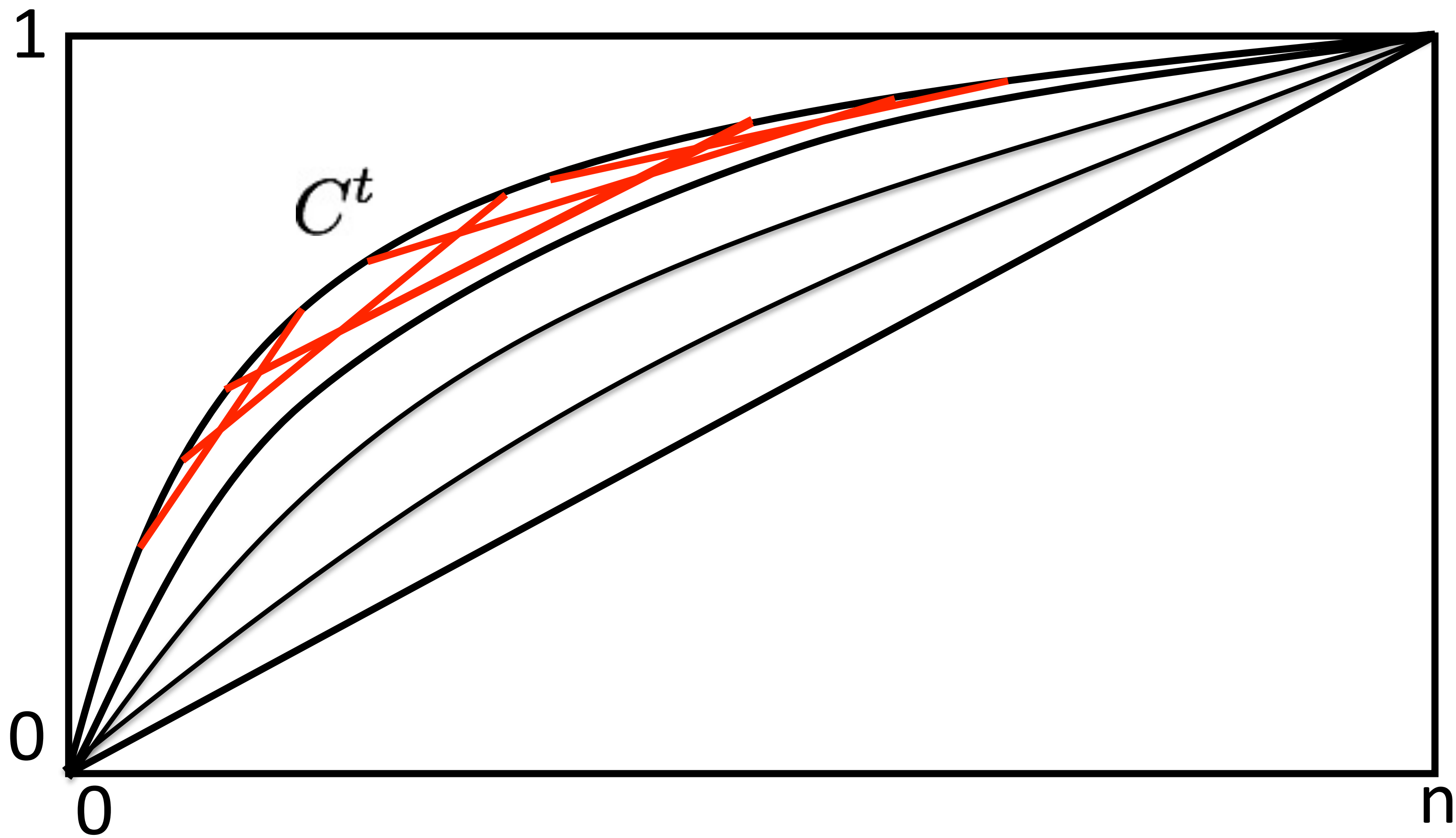
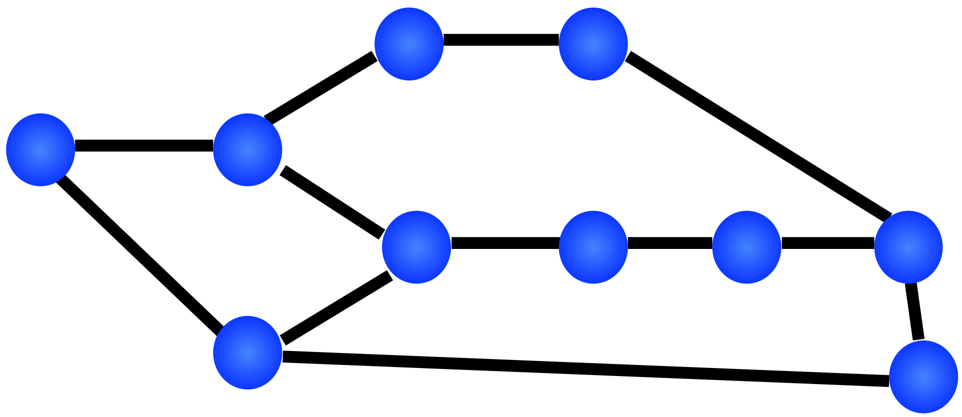


Low-Stretch Spanning Trees

[Alon-Karp-Peleg-West '91]



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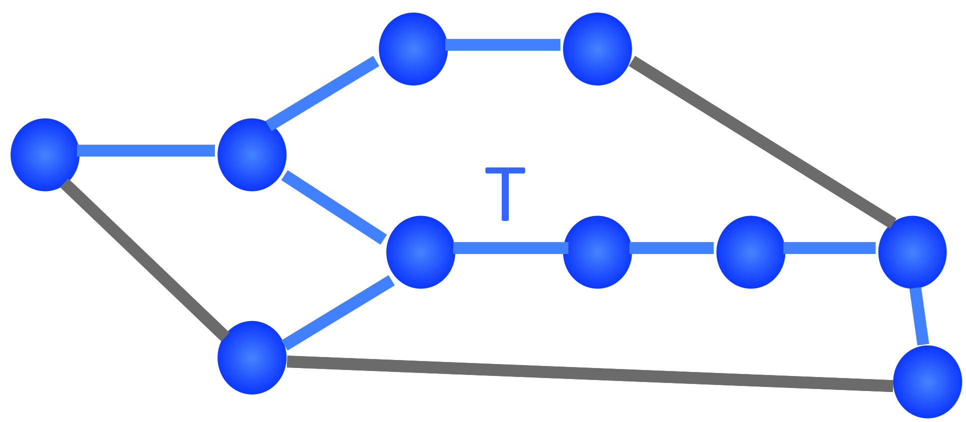
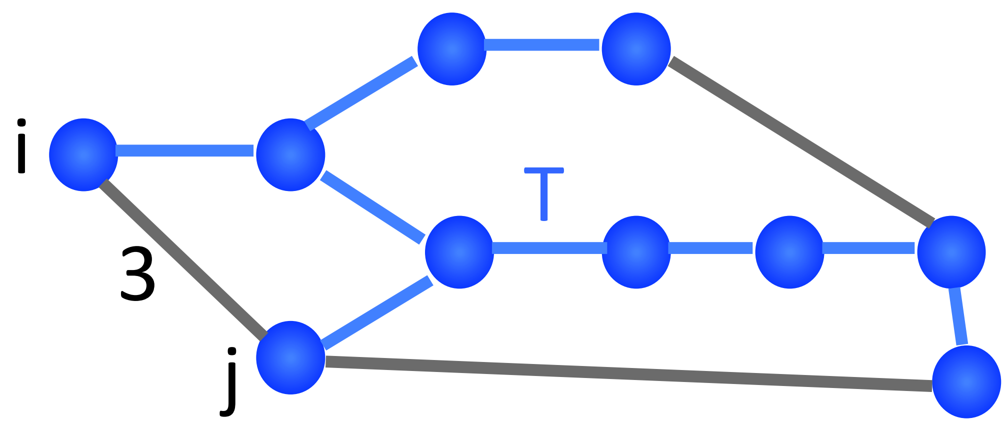


Low-Stretch Spanning Trees (unweighted case)

[Alon-Karp-Peleg-West '91]



$$\mathrm{stretch}_T(i,j) = \mathrm{dist}_T(i,j)$$

Low-Stretch Spanning Trees (unweighted case)

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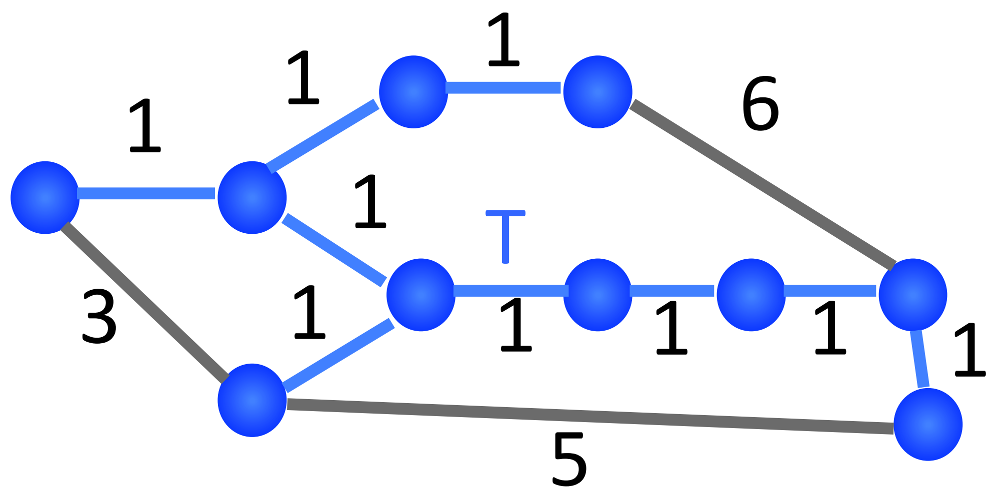

$$\mathrm{stretch}_T(G) = \sum_{(i,j)\in G} \mathrm{dist}_T(i,j)$$

Low-Stretch Spanning Trees

[Alon-Karp-Peleg-West ‘91]

$$\mathrm{stretch}_T(G) = \sum_{(i,j) \in G} \mathrm{dist}_T(i,j)/\mathrm{length}(i,j)$$

Every graph has a spanning tree of low stretch

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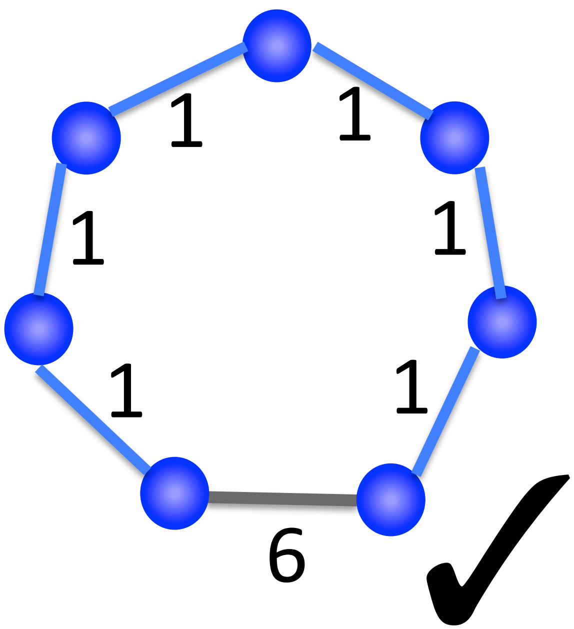

Expander:

Low-Stretch Spanning Trees

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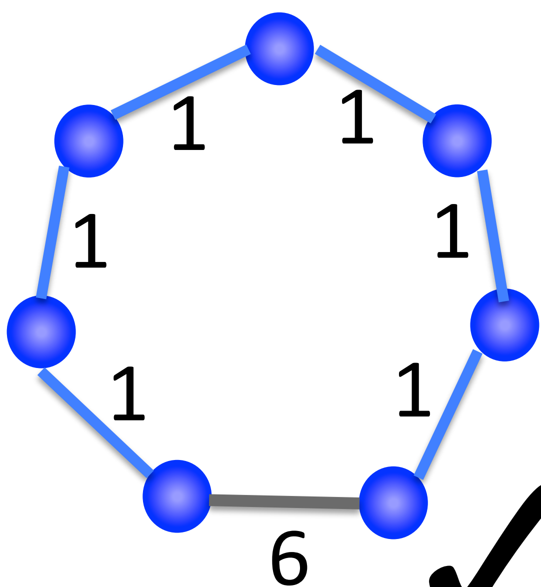

✓

Expander:
low diameter
use shortest path tree

✓

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[Alon-Karp-Peleg-West '91]

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$$\leq m 2^{O(\sqrt{\log n \log\log n})}$$

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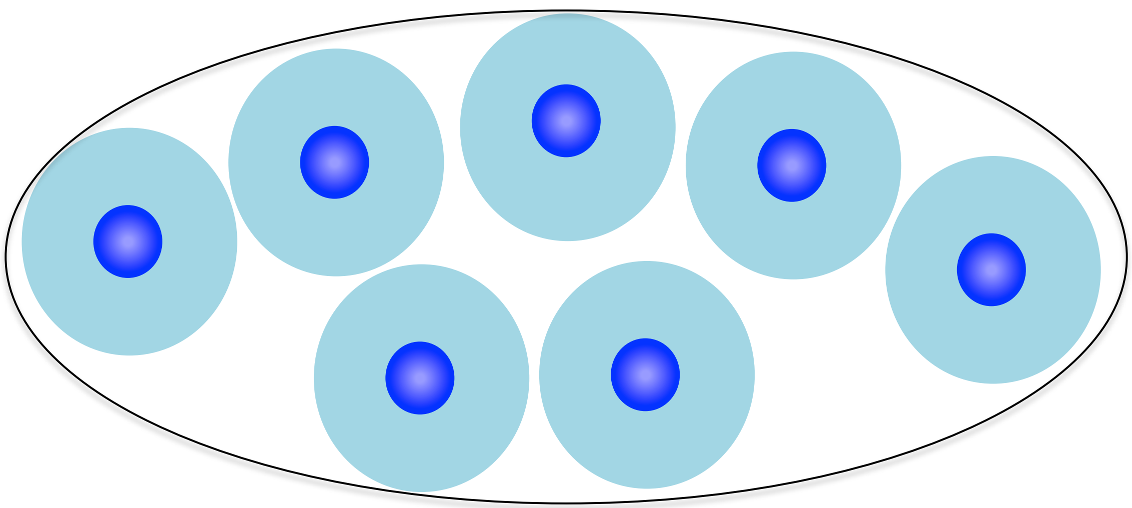
Proof: by BFS/shortest paths.
Grow balls until low boundary (Awerbuch)



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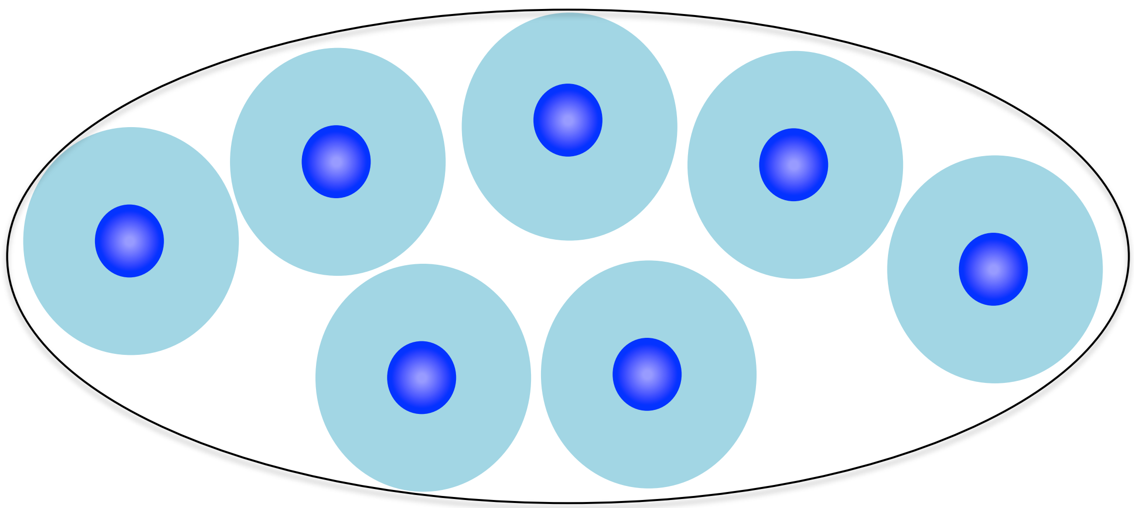

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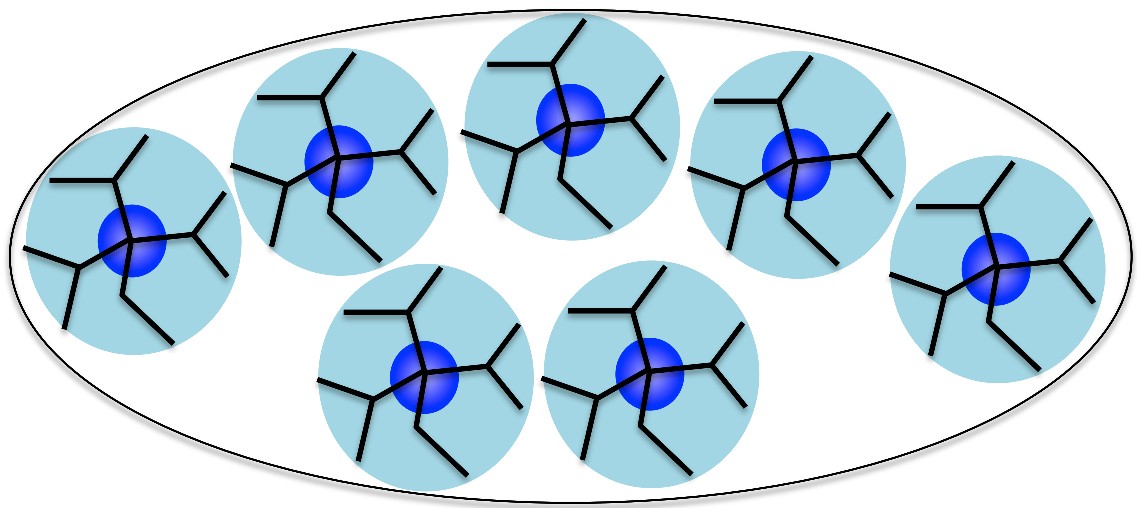

Use shortest path trees inside balls.

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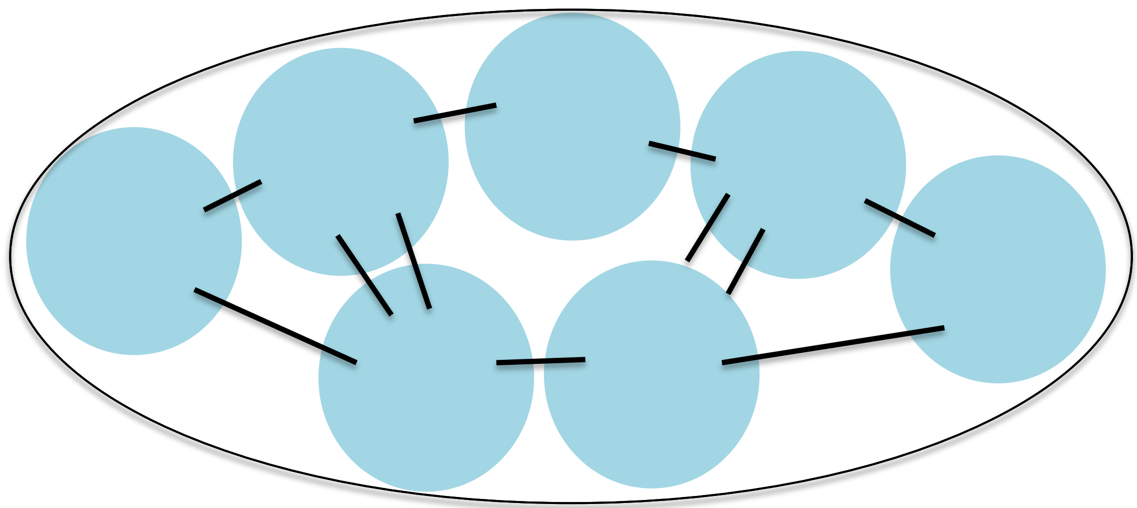

Use shortest path trees inside balls.

Contract balls.

Lower-Stretch Spanning Trees

[Elkin-Emek-S-Teng ‘04, Abraham-Bartal-Neiman ‘08]

Proof: BFS cones centered on a BFS ball.

$$\leq O(m \ \log n \ \log\log n \ (\log\log\log n)^2)$$

Lower-Stretch Spanning Trees

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Proof: BFS cones centered on a BFS ball.

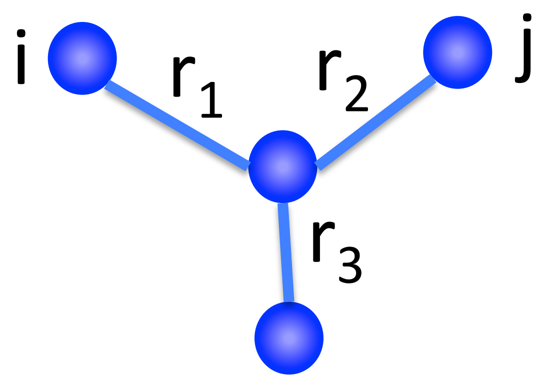
$$\leq O(m \ \log n \ \log\log n \ (\log\log\log n)^2)$$

Can we get $m \log_2 n$?

Algebraic computation of stretch [S-Woo '09]

$$\mathrm{stretch}_T(G) = \mathrm{Trace}\left[L_T^{+} L_G\right]$$

Key: in trees, resistance acts like length.



$$R_{eff}(i,j) = r_1 + r_2$$

Algebraic computation of stretch [S-Woo '09]

$$\mathrm{stretch}_T(G) = \mathrm{Trace}\left[L_T^+ L_G\right]$$

$$\begin{aligned}
\mathrm{Trace}\left[L_T^+ L_G\right] &= \mathrm{Trace}\left[L_T^+ \sum w_{i,j}(e_i - e_j)(e_i - e_j)^T\right] \\
&= \sum w_{i,j}\,\mathrm{Trace}\left[L_T^+ (e_i - e_j)(e_i - e_j)^T\right] \\
&= \sum w_{i,j}\,\mathrm{Trace}\left[(e_i - e_j)^T L_T^+ (e_i - e_j)\right] \\
&= \sum w_{i,j}(e_i - e_j)^T L_T^+ (e_i - e_j) \\
&= \sum w_{i,j} R_T^{eff}(i,j) \\
&= \sum w_{i,j}\,\mathrm{dist}_T(i,j) \\
&= \sum \mathrm{dist}_T(i,j)/\mathrm{length}(i,j)
\end{aligned}$$

Preconditioning: solve in time O(m k)

Need H $\subseteq$ G, H is a tree + m/k edges
such that $\lambda_{max}(L_H^+ L_G) \leq k^2$

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For low-stretch T,

$$\sum_i \lambda_i(L_T^+ L_G) = \text{Trace}\left[L_T^+ L_G\right] \leq O(m \log n)$$

At most s eigenvalues larger than $O((m/s) \log n)$

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Kolla-Makarychev-Saberi-Teng '10:
Fix s eigenvalues with O(s) edges

s = m / log n   k = O(log n)

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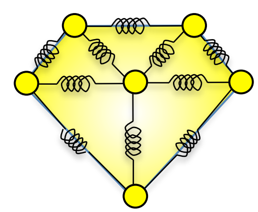
Koutis-Miller-Peng '10:



$k = O(\log^2 n)$

Easy to choose edges: sample by stretch

Faster Laplacian Solvers?

KMST '09 says O(m log n) might be possible

Very powerful primitive

Like sorting

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Better than BFS?

Uses for better graph exploration?

Faster local clustering?

Convergence of processes on graphs?

Conclusion

We’ve produced a lot of gems