Spectral Graph Theory	Lecture 25
Planar Graphs, part 1	
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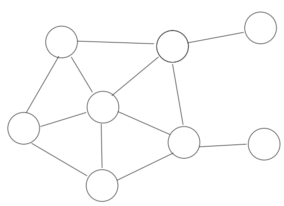
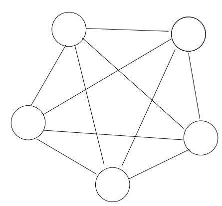
25.1 Introduction

In this and the next lecture, we will explore spectral properties of planar graphs. I'll also spend some time telling you other important facts about planar graphs. Planar graphs relate to some of the most exciting parts of graph theory, and it would be a shame for you not to know something about them.

A graph is planar if it can be drawn in the plane without any crossing edges. That is, each vertex is located at one point of the plane, and a curve from one point to another is drawn between the points corresponding to vertices connected by an edge. None of these curves interesect each other, and may only touch at the points representing the vertices at which they start or end.



(a) A planar graph



(b) A non-planar graph

These figures suggest that even more is true. Each vertex can be represented by a circle, such that the interiors of all these circles are disjoint. The edges can be represented by totally disjoint line segments connecting the boundaries of the circles of which they are endpoints.

Planar graphs originated with the studies of polytopes and of maps. The skeleton (edges) of a three-dimensional polytope provide a planar graph. We obtain a planar graph from a map by representing countries by vertices, and placing edges between countries that touch each other. Assuming each country is contiguous, this gives a planar graph. While planar graphs were introduced for practical reasons, they posess many remarkable mathematical properties. This is one reason we will study them.

A more practical reason for studying planar graphs is that they, and their relatives, appear in many practical applications. The study of two-dimensional images often results in problems related to planar graphs, as does the solution of many problems on the two-dimensional surface of our earth. Many natural three-dimensional graphs arise in scientific and engineering problems. These often come from well-shaped meshes, which share many properties with planar graphs.

25.2 Planar Graphs

Here's a formal definition of a planar graph.

Definition 25.2.1. *A graph is planar if there exists an embedding of the vertices in* $\mathbb{R}^2$*,* $f : V \to \mathbb{R}^2$ *and a mapping of edges* $e \in E$ *to simple curves in* $\mathbb{R}^2$*,* $f_e : [0, 1] \to \mathbb{R}^2$ *such that the endpoints of the curves are the vertices at the endpoints of the edge, and no two curves intersect except possibly at their endpoints.*

Definition 25.2.2. *A graph is planar if there exists an embedding of the vertices in* $\mathbb{R}^2$*,* $f : V \to \mathbb{R}^2$ *such that for all pairs of edges* (a, b) *and* (c, d) *in* E*, with* a*,* b*,* c*, and* d *distinct, the line segment from* $f(a)$ *to* $f(b)$ *does not cross the line segment from* $f(c)$ *to* $f(d)$*.*

It is a theorem that these two definitions are equivalent.

There are many things that you should know about planar graphs. Given an embedding of a planar graph in the plane, we call every region of the plane that is connected when the edges are removed a face. The outside is also a face.

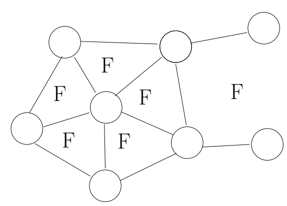


Figure 25.1: In this figure, each face has been marked with an "F".

Theorem 25.2.3 (Euler's Theorem)**.** *Let* $G = (V, E)$ *be a connected planar graph, and let* F *be the set of faces of* G *in some planar embedding. Then,*

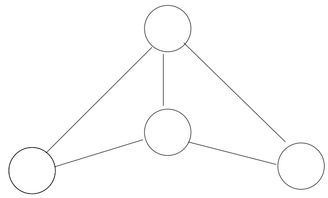
$$|V| - |E| + |F| = 2.$$

This is a wonderful theorem. You can prove it by induction on the number of vertices and edegs. Verify that it holds for the graph with 1 vertex, 1 face, and no edges. Then, see what happens when you add edges and vertices.

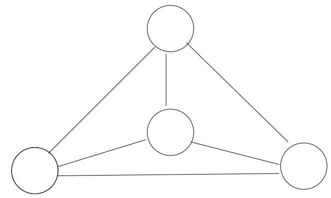
If we place a few restrictions on a planar graph, then the faces become fixed. Recall that a graph is said to be k-vertex-connected if after the removal of any set of fewer than k vertices the graph remains connected. In a planar graph that is 3-vertex-connected, the faces are fixed.

If you add as many edges as possible to a planar graph, subject to its remaining planar, you obtain a graph in which every face is a triangle. The resulting graph is called a "fully triangulated planar graph". By combining Euler's theorem with simple counting, you can prove an upper bound on the number of edges in a fully triangulated planar graph.

Corollary 25.2.4. *If* G *is a fully-triangulated planar graph with* $n \geq 3$ *vertices, then it has* $m = 3n - 6$ *edges. Moreover, every planar graph with* $n \geq 3$ *vertices has at most* $3n - 6$ *edges.*



(a) A planar graph



(b) A fully triangulated planar graph

This enables us to prove that the complete graph on five vertices is not planar, as it has 5 vertices and 10 edges.

25.3 Planar Separators

The famous Planar Separator Theorem of Lipton and Tarjan [LT79] tells us that it is possible to remove $O(\sqrt{n})$ vertices of a planar graph so that no component of the remaining graph has more than $2n/3$ vertices. We will prove a result which shows that eigenvectors can be used to find such separators (after a little work that I will not discuss here).

Theorem 25.3.1 ([ST07])**.** *Let G be a planar graph with n vertices of maximum degree d, and let λ_2 be the second-smallest eigenvalue of its Laplacian. Then,*

$$\lambda_2 \leq \frac{8d}{n}.$$

This theorem helps explain the success of spectral partitioning methods. The proof will involve almost no calculation, but will use some special properties of planar graphs. However, this proof has been generalized to many planar-like graphs, including the graphs of well-shaped 3d meshes [ST07]. This theorem has been extended to graphs of bounded genus by Kelner [Kel06], to graph excluding bounded minors [BLR10]. Bounds on the higher eigenvalues have been obtained in [KLPT09].

25.4 Geometric Embeddings

We typically upper bound λ_2 by evidencing a test vector. Here, we will upper bound λ_2 by evidencing a test embedding. The bound we apply is:

Lemma 25.4.1. *For any $d \geq 1$,*

$$\lambda_2 = \min_{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n \in \mathbb{R}^d : \sum \boldsymbol{v}_i = \boldsymbol{0}} \frac{\sum_{(i,j)\in E} \|\boldsymbol{v}_i - \boldsymbol{v}_j\|^2}{\sum_i \|\boldsymbol{v}_i\|^2}. \tag{25.1}$$

Proof. Let $\boldsymbol{v}_i = (x_i, y_i, \ldots, z_i)$. We note that

$$\sum_{(i,j)\in E} \|\boldsymbol{v}_i - \boldsymbol{v}_j\|^2 = \sum_{(i,j)\in E} (x_i - x_j)^2 + \sum_{(i,j)\in E} (y_i - y_j)^2 + \cdots + \sum_{(i,j)\in E} (z_i - z_j)^2.$$

Similarly,

$$\sum_i \|\boldsymbol{v}_i\|^2 = \sum_i x_i^2 + \sum_i y_i^2 + \cdots + \sum_i z_i^2.$$

It is now trivial to show that $\lambda_2 \geq RHS$: just let $x_i = y_i = \cdots = z_i$ be given by an eigenvector of λ_2. To show that $\lambda_2 \leq RHS$, we apply my favorite inequality: $\frac{A+B+\cdots+C}{A'+B'+\cdots+C'} \geq \min\left(\frac{A}{A'}, \frac{B}{B'}, \ldots, \frac{C}{C'}\right)$, and then recall that $\sum x_i = 0$ implies

$$\frac{\sum_{(i,j)\in E}(x_i - x_j)^2}{\sum_i x_i^2} \geq \lambda_2.$$

□

For an example, consider the natural embedding of the square with corners $(\pm 1, \pm 1)$.

The key to applying this embedding lemma is to obtain the right embedding of a planar graph. Usually, the right embedding of a planar graph is given by Koebe's embedding theorem, which I will now explain. I begin by considering one way of generating planar graphs. Consider a set of circles $\{C_1, \ldots, C_n\}$ in the plane such that no pair of circles intersects in their interiors. Associate a vertex with each circle, and create an edge between each pair of circles that meet at a boundary. The resulting graph is clearly planar. Koebe's embedding theorem says that *every planar graph results from such an embedding.*

Theorem 25.4.2 (Koebe)**.** *Let $G = (V, E)$ be a planar graph. Then there exists a set of circles $\{C_1, \ldots, C_n\}$ in $\mathbb{R}^2$ that are interior-disjoint such that circle C_i touches circle C_j if and only if $(i, j) \in E$.*

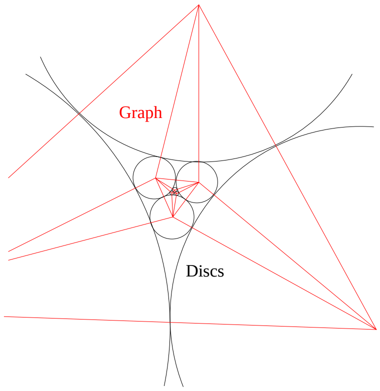
This is an amazing theorem, which I won't prove today. You can find a proof at

`http://math.mit.edu/~spielman/course/lect3.html`

Such an embedding is often called a *kissing disk* embedding of the graph. From a kissing disk embedding, we obtain a natural choice of $\boldsymbol{v}_i$: the center of disk C_i. Let r_i denote the radius of this disk. We now have an easy upper bound on the numerator of (25.1): $\|\boldsymbol{v}_i - \boldsymbol{v}_j\|^2 = (r_i + r_j)^2 \leq 2r_i^2 + 2r_j^2$. On the other hand, it is trickier to obtain a lower bound on $\sum \|\boldsymbol{v}_i\|^2$. In fact, there are graphs whose kissing disk embeddings result in

$$(25.1) = \Theta(1).$$

These graphs come from triangles inside triangles inside triangles... Such a graph is depicted below:



We will fix this problem by lifting the planar embeddings to the sphere by stereographic projection. Given a plane, $\mathbb{R}^2$, and a sphere S tangent to the plane, we can define the stereographic projection map, Π, from the plane to the sphere as follows: let $\boldsymbol{s}$ denote the point where the sphere touches the plane, and let $\boldsymbol{n}$ denote the opposite point on the sphere. For any point $\boldsymbol{x}$ on the plane, consider the line from $\boldsymbol{x}$ to $\boldsymbol{n}$. It will intersect the sphere somewhere. We let this point of intersection be $\Pi(\boldsymbol{x})$.

The fundamental fact that we will exploit about stereographic projection is that *it maps circles to circles!* So, by applying stereographic projection to a kissing disk embedding of a graph in the plane, we obtain a kissing disk embedding of that graph on the sphere. Let $D_i = \Pi(C_i)$ denote the image of circle C_i on the sphere. We will now let $\boldsymbol{v}_i$ denote the center of D_i, on the sphere.

If we had $\sum_i \boldsymbol{v}_i = \boldsymbol{0}$, the rest of the computation would be easy. For each i, $\|\boldsymbol{v}_i\| = 1$, so the denominator of (25.1) is n. Let r_i denote the straight-line distance from $\boldsymbol{v}_i$ to the boundary of D_i. We then have

$$\|\boldsymbol{v}_i - \boldsymbol{v}_j\|^2 \leq (r_i + r_j)^2 \leq 2r_i^2 + 2r_j^2.$$

So, the denominator of (25.1) is at most $2d \sum_i r_i^2$. On the other hand, the area of the cap encircled by D_i is at least πr_i^2. As the caps are disjoint, we have

$$\sum_i \pi r_i^2 \leq 4\pi,$$

which implies that the denominator of (25.1) is at most

$$2d \sum_i r_i^2 \leq 8d.$$

Putting these inequalities together, we see that

$$\min_{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n \in \mathbb{R}^d : \sum \boldsymbol{v}_i = \boldsymbol{0}} \frac{\sum_{(i,j)\in E} \|\boldsymbol{v}_i - \boldsymbol{v}_j\|^2}{\sum_i \|\boldsymbol{v}_i\|^2.} \leq \frac{8d}{n}.$$

Thus, we merely need to verify that we can ensure that

$$\sum_i \boldsymbol{v}_i = \boldsymbol{0}. \qquad (25.2)$$

Note that there is enough freedom in our construction to believe that we could prove such a thing: we can put the sphere anywhere on the plane, and we could even scale the image in the plane before placing the sphere. By carefully combining these two operations, it is clear that we can place the center of gravity of the $\boldsymbol{v}_i$s close to any point on the boundary of the sphere. It turns out that this is sufficient to prove that we can place it at the origin.

I think I'll skip the details of this argument, and tell you other things about planar graphs instead.

References

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