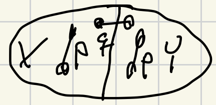
2025-Feb-10 Let P be symmetric with entries in $[0,1]$, zero diagonal

$$M(a,b) = \begin{cases} 1 & \text{prob } P(a,b) \\ 0 & \text{o.w.} \end{cases} \qquad R = M - P$$

If $\|R\|$ small, eigvals and eigvecs of M and P are similar.

So, if eigvecs of P are useful, those of M could be, too.

Stochastic Block Model / Planted partition  $p > q$ $\quad P = \begin{pmatrix} pJ_{n/2} & qJ_{n/2} \\ qJ_{n/2} & pJ_{n/2} \end{pmatrix} - pI_n$

Can you recover X and Y from M?

For any matrix M, $\|M\| = \max_{\|x\|=1} \|Mx\|$ = largest singular value. Symmetric $= \max_{\|x\|=1} x^T M x$

Perturbation Theory. If $A - B = R$, A has eigvals $\alpha_1 \geq \cdots \geq \alpha_n$, B has eigvals $\beta_1 \geq \cdots \geq \beta_n$

$|\alpha_i - \beta_i| \leq \|R\|$. If α_i and β_i have multiplicity 1 and ϕ_i and ψ_i are their eigvecs,

and $\theta = \text{angle}(\phi_i, \psi_i)$ then $\sin 2\theta \leq \dfrac{2\|R\|}{\min(\alpha_{i-1} - \alpha_i, \alpha_i - \alpha_{i+1})}$. Davis & Kahan

Why $\min_j |\alpha_i - \alpha_j|$ matters Ex. $A = \begin{pmatrix} 1+\varepsilon & 0 \\ 0 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ 0 & 1+\varepsilon \end{pmatrix}$ angle change $\frac{\pi}{2}$

or $B = \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{bmatrix}$ $\psi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

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proof (that $\sin\theta \leq 2R / \min_{j \neq i} |\alpha_i - \alpha_j|$). Assume wlog $\alpha_i = 0$, by $A \to A - \alpha_i I$, $B \to B - \alpha_i I$

$\|A\psi_i\| = \|(B+R)\psi_i\| \leq \|B\psi_i\| + \|R\psi_i\|$ (assuming $\|\psi_i\| = 1$) $\leq \beta_i + \|R\| \leq 2\|R\|$

Set $c_j = \phi_i^T \psi_j$. $\cos^2\theta = c_j^2$, $\sin^2\theta = 1 - c_j^2$. Let $\delta = \min_{j \neq i} |\alpha_i - \alpha_j|$

$$\|A\psi_i\|^2 = \sum_j c_j^2 \alpha_j^2 = \sum_{j \neq i} c_j^2 \alpha_j^2 \geq \delta^2 \sum_{j \neq i} c_j^2 = \delta^2 \sin^2\theta$$

So, $\delta \sin\theta \leq 2\|R\|$, $\sin\theta \leq \dfrac{2\|R\|}{\delta}$

Given $|X| = \frac{n}{2}$, $|Y| = \frac{n}{2}$, edges inside X, Y prob p, between prob q, $p > q$ recover X, Y.

Let $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ be eigvals of $P + pI = \begin{pmatrix} p J_{n/2} & q J_{n/2} \\ q J_{n/2} & p J_{n/2} \end{pmatrix}$ $\alpha_1 = \frac{n}{2}(p+q)$ $\phi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ rest 0

$\alpha_2 = \frac{n}{2}(p-q)$ $\phi_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

We want $p - q \leq (p+q-(p-q))$, so $p \leq 3q$ and $\min_{i \neq 2} |\alpha_2 - \alpha_i| = \alpha_2$

Thm (using technique from last lecture). For $R = M - P$, $p \geq \frac{c(\log n)^4}{n}$, whp $\|R\| \leq 3\sqrt{p(1-p)n}$.

So, $\sin \angle(\phi_2, \psi_2) \leq 2 \frac{3\sqrt{p(1-p)n}}{\frac{n}{2}(p-q)} = \frac{12}{\sqrt{n}} \frac{\sqrt{p(1-p)}}{p-q}$ $\phi_2(a) = \begin{cases} 1/\sqrt{n} & a \in X \\ -1/\sqrt{n} & a \in Y \end{cases}$

Let $S = \{a : \psi_2(a)\phi_2(a) < 0\}$ for such a, $|\phi_2(a) - \psi_2(a)| \geq \frac{1}{\sqrt{n}}$,

so $\|\phi_2 - \psi_2\|^2 \geq \frac{|S|}{n}$ As $\|\phi_2 - \psi_2\| \leq \sqrt{2}\sin\theta$, $\frac{|S|}{n} \leq 2(\sin\theta)^2$

$|S| \leq 2n \frac{144}{n} \frac{p(1-p)}{(p-q)^2} = 288 \frac{p(1-p)}{(p-q)^2}$ if $p > q$ const, get constant # errors

Consider $p = \frac{1}{2}$, $q = \frac{1}{2} - \frac{17}{\sqrt{n}}$ gives $|S| \leq n/4$.
