2025-Feb-5

Random Graphs. Erdős-Rényi model $G(n,p)$: n-vertex graph. Each edge with prob p, independently

(Think $p = 1/2$). $M(a,b) = \begin{cases} 1 & \text{prob } p \\ 0 & \text{prob } 1-p \end{cases}$ $\quad M(a,b) = M(b,a)$. $M(a,a) = 0$ $\quad \mathbb{E}[M(a,b)] = p$

$\mathbb{E}[M] = p(J - I)$, recall J = all-1 matrix.

let $R = M - p(J-I)$. Will show $\|R\|$ probably small. So, $\text{eigs}(M) \approx \text{eigs}(p(J-I))$

$\text{eigs } p(J-I) = p \cdot \text{eigs}(J-I) = p(n-1)$, and $-p$ with mult $n-1$. $\boxed{\text{Jupyter}}$

$R(a,b) = \begin{cases} 1-p & \text{prob } p \\ -p & \text{prob } 1-p \end{cases}$ $\quad$ So $\mathbb{E}\, R(a,b) = 0$. $\quad \|R\| = \max\limits_{\|u\|=1} |u^T R u|$

__Thm 1__ For $p = 1/2$, $\Pr[\|R\| \geq t] \leq e^{-(t/2)^2} \sqrt{\pi}\, n\, 2^{n-1}$. If $t \gg 2\sqrt{\ln 2}\sqrt{n}$, then

$e^{-(t/2)^2} \ll 2^{-n}$, and $\longrightarrow$ is small. $\quad 2\sqrt{\ln 2} < 5/3$

__Lem 1__ For a fixed $\|u\| = 1$, $\Pr\limits_R[|u^T R u| > t] \leq 2e^{-t^2}$

__Hoeffding's Ineq__ Let $X_1, \ldots, X_m$ be indep random variables such that

$\mathbb{E}\, X_i = 0$ and $X_i \in [\alpha_i, \beta_i]$. Then, $\forall t > 0$, $\Pr[\sum X_i > t] \leq e^{-\frac{2t^2}{\sum_i (\beta_i - \alpha_i)^2}}$

__proof of Lem 1__ Random variables $X_{a,b} = 2R(a,b)u(a)u(b)$. $u^T R u = \sum\limits_{a<b} 2R(a,b)u(a)u(b) = \sum\limits_{a<b} X_{a,b}$

$\alpha_{a,b} = -u(a)u(b)$ $\quad \sum\limits_{a<b} (\beta_{a,b} - \alpha_{a,b})^2 = \sum\limits_{a<b} 4\, u(a)^2 u(b)^2 = 2\sum\sum\limits_{a \neq b} u(a)^2 u(b)^2 \leq 2\sum\limits_a u(a)^2 \sum\limits_b u(b)^2 \leq 2$

$\beta_{a,b} = u(a)u(b)$

So, $\Pr\limits_R[u^T R u \geq t] = \Pr[\sum\limits_{a<b} X_{a,b} \geq t] \leq e^{-2t^2/2} = e^{-t^2}$

$\Pr[-u^T R u \geq t] \leq e^{-t^2}$, so $2e^{-t^2}$

Lem2 For a fixed R, $\Pr_{\|u\|=1}[\, u^T R u \ge \frac{1}{2}\|R\| \,] \ge \frac{1}{\sqrt{\pi}\, n\, 2^{n-1}}$

proof of Thm 1 $(u^T R u \ge \frac{1}{2}\|R\|)$ and $(\|R\| \ge t) \Rightarrow u^T R u \ge t/2$

So, $\Pr_{\substack{\|u\|=1 \\ R}}[\, u^T R u \ge \frac{1}{2}\|R\| \text{ and } \|R\| \ge t \,] = \Pr_R[\, \|R\| \ge t \,] \Pr_u[\, u^T R u \ge \frac{1}{2}\|R\| \mid \|R\| \ge t \,] \le \Pr[\, u^T R u \ge t/2 \,]$

$\rightarrow \Pr_R[\, \|R\| \ge t \,] \le \Pr[\, u^T R u \ge t/2 \,] \Big/ \Pr_u[\, u^T R u \ge \frac{1}{2}\|R\| \mid R \,] \le 2e^{-(t/2)^2} \sqrt{\pi}\, n\, 2^{n-1}$

Lem 3 let $R\psi = \|R\|\psi$, $\|\psi\| = 1$. If $u^T\psi \ge \sqrt{3}/2$ then $|u^T R u| \ge \frac{1}{2}\|R\|$

proof. let eigs$(R) = \rho_1 \ge \cdots \ge \rho_n$. wlog $\|R\| = \rho_1 \ge |\rho_n|$

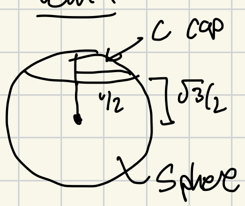
let $u = \sum_i c_i \psi_i$, $c_i = \psi_i^T u$, $c_1 \ge \sqrt{3}/2$, $\sum c_i^2 = 1$

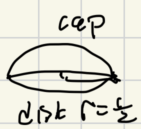
$\rightarrow u^T R u = \sum_i c_i^2 \rho_i \ge c_1^2 \rho_1 - \sum_{i \ge 2} c_i^2 \rho_1 = \rho_1 (c_1^2 - \sum_{i \ge 2} c_i^2) = \rho_1 (2c_1^2 - 1) \ge \rho_1/2$

Lem 4 For any $\|\psi\| = 1$, $\Pr_u[\, u^T\psi \ge \sqrt{3}/2 \,] \ge 1/(\sqrt{\pi}\, n\, 2^{n-1})$

Lem 4 + Lem 3 $\Rightarrow$ Lem 2.

Lem 4 $\Pr_u[\, u^T\psi \ge \sqrt{3}/2 \,] = \frac{\text{surf(cap)}}{\text{surf(sphere)}} \ge \frac{\text{surf(disk)}}{\text{surf(sphere)}} = \frac{(\frac{1}{2})^{n-1} \pi^{\frac{n-1}{2}} / \Gamma(\frac{n-1}{2}+1)}{n\pi^{n/2} / \Gamma(\frac{n}{2}+1)}$





surf(cap) ≥ surf(Disk)

$\Gamma(n+1) = n!$ is increasing

$$= \frac{2^{-(n-1)}\, \Gamma(\frac{n}{2}+1)}{\sqrt{\pi}\, n\, \Gamma(\frac{n-1}{2}+1)} \ge \frac{1}{\sqrt{\pi}\, n\, 2^{(n-1)}}$$

Understand shape of histogram of $\rho_1, \dots, \rho_n$ by moments $\sum \rho_i^{\ell} = Tr(R^{\ell})$, ℓ even

$\|R\|^{\ell} = \max_i \rho_i^{\ell} \le Tr(R^{\ell}) \to \|R\| \le (Tr(R^{\ell}))^{1/\ell}$

Can prove, for $\ell^8 \le \frac{1}{2} np(1-p)$, $\mathbb{E}\, Tr(R^{\ell}) \le 2n(2\sqrt{np(1-p)})^{\ell} \triangleq u^{\ell}$ $\quad u \approx \sqrt{n}$ for $p = \frac{1}{2}$

So, for $\varepsilon > 0$, $Pr[\|R\| \ge (1+\varepsilon)u] = Pr[\|R^{\ell}\| \ge (1+\varepsilon)^{\ell} u^{\ell}] \le Pr[Tr(R^{\ell}) \ge (1+\varepsilon)^{\ell} u^{\ell}]$

$\le Pr[Tr(R^{\ell}) \ge (1+\varepsilon)^{\ell} \mathbb{E}\, Tr R^{\ell}] \le \frac{1}{(1+\varepsilon)^{\ell}} \approx e^{-\varepsilon \ell}$

$Tr(R^{\ell}) = \sum_{a_0} R^{\ell}(a_0, a_0)$

$\mathbb{E}\, R^{\ell}(a_0, a_0) = \sum_{a_1} R(a_0, a_1) R^{\ell-1}(a_1, a_0) = \mathbb{E} \sum_{a_1, a_2, \dots, a_{\ell-1}} R(a_0, a_{\ell-1}) \prod_{i=1}^{\ell-1} R(a_{i-1}, a_i)$

For indep X, Y $\mathbb{E}[XY] = (\mathbb{E}X)(\mathbb{E}Y)$. Here $\mathbb{E}\, R(a_{i-1}, a_i) = 0$.

$\rightarrow$ only nonzero when each term appears at least twice

Let $\{b_j, c_j\}_j$ be the distinct consecutive pairs in $\{a_{i-1}, a_i\}_{1 \le i < \ell} \cup \{a_0, a_\ell\}$

let $\{b_j, c_j\}$ appear d_j times. Then

$\mathbb{E}\, R(a_0, a_{\ell-1}) \prod_{i=1}^{\ell-1} R(a_{i-1}, a_i) = \prod_j \mathbb{E}\, R(b_j, c_j)^{d_j} \le \prod_j p(1-p)$

$\mathbb{E}\, R^{\ell}(a_0, a_0) =$ Sum over sequences $a_1 \dots a_{\ell-1}$ s.t. each pair occurs ≥ 2 times of $(p(1-p))^{\#\text{distinct pairs}}$

$\#$ distinct pairs $\ge$ $\#$ distinct elements $\qquad W_{n,\ell,k} = \#$ sequences $a_1, \dots, a_{\ell-1}$ with k distinct elements

$\mathbb{E}\, R^{\ell}(a_0, a_0) \le \sum_{k=1}^{\ell/2} (p(1-p))^k W_{n,\ell,k}$ $\quad$ need to count $W_{n,\ell,k}$

Crude bound $W_{n,\ell,k} \le \ell^{\ell} n^k$ $\quad$ enough for $\mathbb{E}\, Tr(R^{\ell}) \le n \cdot n^{\ell/2} \ell^{\ell}$

$\downarrow$

2^{ℓ} if careful $\qquad u \approx \sqrt{n}\, n^{1/\ell} \ell \quad \ell = \lg n.$

Pick the location each distinct element first appears $\binom{\ell}{k} \le 2^{\ell}$ $\qquad 2^{\ell} (\frac{\ell}{2})^{\ell} n^k \le \ell^{\ell} n^k$

Pick identies of the distinct elements $\le n^k$

For every repeat, which of the k it was $\le k^{\ell-k} \le (\frac{\ell}{2})^{\ell}$