2025 - Feb - 3 Comparing Graphs, lower bounds on λ_2

$A \succeq 0$ means A is positive semi-definite = symmetric, no negative eigvals

$\Leftrightarrow x^T A x \geq 0 \ \forall x$

$A \succeq B$ if $A - B \succeq 0$. Is a partial order. $A \succeq B$ and $B \succeq C \Rightarrow A \succeq C$

but not all comparable. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \not\succeq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

For all symmetric C, $A \succeq B \rightarrow A + C \succeq B + C$

Overload $G \succeq H$ means $L_G \succeq L_H$. Recall $L_G = \sum_{a \sim b} w_{a,b} (x(a) - x(b))^2$

So, if H has weights $z_{a,b}$ with $w_{a,b} \geq z_{a,b} \ \forall a, b$ then $G \succeq H$

We often write inequalities like $G \succeq cH$, for some $c \geq 0$. cH = edge weights multiplied by c

cH is graph such that $L_{cH} = cL_H$. Equiv to $\frac{1}{c} G \succeq H$

Prop 1 If $G \succeq cH$ then $\lambda_k(G) \geq c \lambda_k(H)$ for all k

proof: $\lambda_k(G) = \min_{\dim(S) = k} \max_{x \in S} \frac{x^T L_G x}{x^T x} \geq \min_{\dim(S) = k} \max_{x \in S} c \cdot \frac{x^T L_H x}{x^T x} = c \lambda_k(H)$

Path Inequality $(n-1) P_n \succeq G_{1,n}$, where $G_{1,n}$ just has edge $(1, n)$

proof Equivalent to $\forall x \in \mathbb{R}^n$ $(n-1) \sum_{a=1}^{n-1} (x(a) - x(a+1))^2 \geq (x(1) - x(n))^2$

Set $\Delta(a) = x(a) - x(a+1)$, so $x(1) - x(n) = \sum_a \Delta(a)$

NTS $(n-1) \sum_{a=1}^{n-1} \Delta(a)^2 \geq \left(\sum_a \Delta(a)\right)^2$. Implied by Cauchy-Schwartz

$\left(\sum_a \Delta(a)\right)^2 = (\mathbb{1}_{n-1}^T \Delta)^2 \geq \|\mathbb{1}_{n-1}\|^2 \|\Delta\|^2 = (n-1) \sum_{1 \leq a < n} \Delta(a)^2$ ✓

Prop 2 $\lambda_2(P_n) \ge \frac{6}{n^2-1}$. Last class learned $\lambda_2(P_n) \approx \frac{\pi^2}{n^2}$, and $\le \frac{12}{n^2+n}$

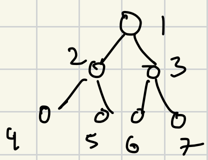
proof Will prove $P_n \succcurlyeq c K_n$, and recall $\lambda_2(K_n) = n$, so $\lambda_2(P_n) \ge cn$

Write $K_n = \sum_{a<b} G_{a,b}$. Let $P_{a,b}$ be path from a to b. $G_{a,b} \preccurlyeq (b-a) P_{a,b}$

$$K_n = \sum_{a<b} G_{a,b} \preccurlyeq \sum_{a<b} (b-a) P_{a,b} \preccurlyeq \sum_{a<b} (b-a) P_n = \left(\sum_{a<b} b-a\right) P_n = \frac{n(n+1)(n-1)}{6} P_n$$

So, $P_n \succcurlyeq \frac{6}{n(n+1)(n-1)} K_n \Rightarrow \lambda_2(P_n) \ge \frac{6}{(n+1)(n-1)}$

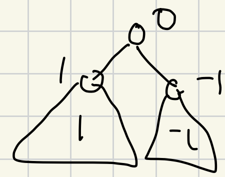
Complete Binary Tree T_n



← depth 2

$n = 2^{d+1} - 1$ $d = 2$, depth

$\lambda_2(T_n) \le \frac{2}{n-1}$



Prop $\lambda_2(T_n) \ge \frac{1}{(n-1)\log_2 n}$ Let $T_{a,b}$ be unique path in T_n from a to b

$T_{a,b}$ has length $\le 2d \le 2\log_2(n)$.

So $K_n = \sum_{a<b} G_{a,b} \le 2d \sum_{a<b} T_{a,b} \le 2d \sum_{a<b} T_n = 2d \binom{n}{2} T_n = dn(n-1) T_n$

$\to T_n \succcurlyeq \frac{1}{n(n-1)\log_2(n)} K_n \to \lambda_2(T_n) \ge \frac{n}{n(n-1)\log_2(n)} = \frac{1}{(n-1)\log_2 n}$

Experiment. $\lambda_2(T_n) \approx \frac{1}{n}$. So, let's improve the lower bound

Wted path inequality. Let P_w be path weight of $(a, a+1)$ be w_a. Then

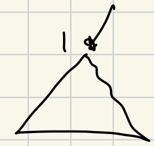
$G_{1,n} \preccurlyeq \left(\sum_a \frac{1}{w_a}\right) P_w$ **proof** $\Delta(a) = (x(a) - x(a+1))^2$.

NTS $\left(\sum_a \Delta(a)\right)^2 \le \sum_a \frac{1}{w_a} \sum_{1 \le a < n} w_a \Delta(a)^2$. Let $\gamma(a) = \Delta(a)\sqrt{w_a}$

$$\left(\sum_a \Delta(a)\right)^2 = \left(\gamma^T w^{-1/2}\right)^2 \le \|\gamma\|^2 \|w^{-1/2}\|^2 = \sum_a \frac{1}{w_a} \sum_a w_a \Delta(a)^2$$

Prop $\lambda_2(T_n) \geq \frac{1}{2n}$. For $a<b$, let $\hat{T}_{a,b}$ be path from a to b in T,

but give edges weight $= 2^{\text{depth}}$, starting edge depths at 1.



So, weights on path are $2, 4, 8, \dots, 2^{d(a)}, 2, 4, 8, \dots, 2^{d(b)}$

$\left(\sum \frac{1}{w_a}\right) = \left(\frac{1}{2}+\frac{1}{4}+\cdots\right) + \left(\frac{1}{2}+\frac{1}{4}+\cdots\right) \leq 2$ $\quad G_{a,b} \preceq 2\hat{T}_{a,b}$

And $\sum_{a<b} \hat{T}_{a,b}$ -- an edge $c \to d$ has weight $2^{d(c)}$, used $\leq \left(2^{d+1-d(c)} - 1\right) n$ times

Total weight $\leq n\left(2^{d+1-d(c)}-1\right) 2^{d(c)} \leq n^2$. So $\sum_{a<b} \hat{T}_{a,b} \preceq n^2 T_n$

$K_n \preceq 2 \sum_{a<b} \hat{T}_{a,b} \preceq 2n^2 T_n \longrightarrow \lambda_2(T_n) \geq \frac{1}{2n}$

Approximations $\quad H \approx_c G \quad$ if $\quad \frac{1}{c} H \preceq G \preceq cH$

ex. let G be random: each edge with prob $\frac{1}{2}$. Then $G \approx_c \frac{1}{2} K_n$ $\; c = 1 + 1/\sqrt{n}$

expanders are d-regular, d const, $1+\varepsilon$ approx of $\frac{d}{n} K_n$

Sparsifier: $1+\varepsilon$ approx of any graph. #edges $\approx n/\varepsilon^2$
