Jan 29, 2025 $\quad \Theta(S) = \frac{|\partial(S)|}{|S|} \geq \lambda_2\left(1 - \frac{|S|}{n}\right)$ left $\qquad \frac{|\partial(S)|}{|S|\,|V-S|}|V| \geq \lambda_2$

$$(Lx)(a) = \sum_{b \sim a} x(a) - x(b) = d(a)x(a) - \sum_{b\sim a} x(b)$$

$\lambda_1 = 0, \quad \psi_1 = \mathbb{1}/\sqrt{n}$

$$\sum \lambda_i = \mathrm{Tr}(L) = \sum_a d(a)$$

$K_n \quad E = \{(a,b) : a \neq b\}$ Has $\lambda_2 = \cdots = \lambda_n = n$

proof 1 Let ψ be s.t. $\mathbb{1}^T\psi = 0$

$(L\psi)(a) = (n-1)\psi(a) - \underbrace{\sum_{b\neq a}\psi(b)}_{=-\psi(a)}$

$= n\psi(a)$

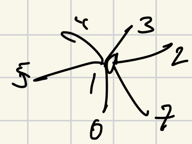
proof 2 $L_{K_n} = nI - J$ all ones $= nI - \mathbb{1}\mathbb{1}^T$

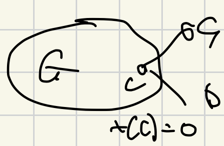
so $\psi^T\mathbb{1} = 0 \to L_{K_n}\psi = (nI - \mathbb{1}\mathbb{1}^T)\psi = n\psi$

For every $S \subset V$, $|\partial(S)| = |S|\cdot|V-S|$

so $\frac{|\partial(S)|}{|S|\,|V-S|}\cdot|V| = n \geq \lambda_2$ is tight.

Star graph S_n. $V = \{1 \ldots n\}$ $E = \{(1,a), a \geq 2\}$





$x(a) = 1$

$x(b) = -1$

$x(c) = 0$

Lem let $G = (V,E)$ be any graph with degree-1 vertices a, b s.t. (a,c) and $(b,c) \in E$

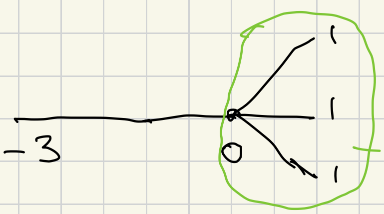
Then $\delta_a - \delta_b$ is eigvec of L_G of eigval 1.

$x(a) - x(c) = x(a)$

$x(b) - x(c) = x(b)$

$2x(c) - x(a) - x(b) = 0 = x(c)$

zero elsewhere.



Fiedler says connected

Span$\{\delta_a - \delta_b : a, b \geq 2\}$ has dim $n-2$. 0 is an eigval. So, need one more call it λ.

$\mathrm{Tr}(L) = 2(n-1) = (n-2) + 0 + \lambda \implies \lambda = n$

For corresp eigvec, $\psi^T(\delta_a - \delta_b) = 0 \;\forall a,b \geq 2$. $\psi^T\mathbb{1} = 0$

$\psi(1) + (n-1)\psi(2) = 0$. Can take $\psi(1) = n-1$, $\psi(2) = -1$

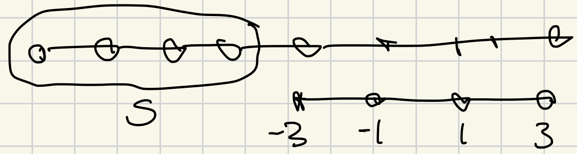
Compare to adj: $\pm\sqrt{n}, 0^{n-2}$

$\lambda_2 = 1$

so $\frac{|\partial(S)|}{|S|\,|V-S|}\cdot|V| \geq \frac{|V|}{|V-S|} \geq 1$

not bad

Path P_n $V=\{1..n\}$ $E=\{(a,a+1) : 1\le a < n\}$



$|S|=n/2$ $|\partial(S)|=1$ $\frac{\partial(S)}{|S|\cdot|V-S|}|V| = \frac{4}{n} \ge \lambda_2$

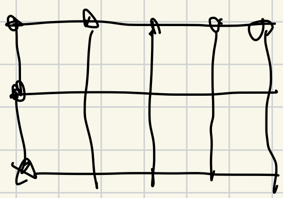
λ_2 much smaller. $x(a)=2a-(n+1)$ $\sum_a x(a) = \left(2\sum_{a=1}^{n} a\right) - n(n+1) = 2\binom{n+1}{2} - n(n+1) = 0$

For $(a,b)\in E$ $(x(a)-x(b))^2 = 2^2 = 4 \rightarrow x^TLx = 4(n-1)$

$x^Tx = \sum_a (x(a))^2 = \sum_a (2a-(n+1))^2 = (n+1)n(n-1)/3$

$\frac{x^TLx}{x^Tx} = \frac{4(n-1)}{(1/3)n(n-1)(n+1)} = \frac{12}{n(n+1)}$

Products Grid



$P_3 \times P_5$

Thm If G has eigvals $\lambda_1..\lambda_m$ eigvecs $\alpha_1..\alpha_m$
H has eigvals $\mu_1..\mu_n$ eigvecs $\beta_1..\beta_n$
$G\times H$ has eigval $\lambda_i+\mu_j$ with eigvec $\gamma_{i,j}(a,b)=\alpha_i(a)\beta_j(b)$

For $G=(V,E)$, $H=(W,F)$ $G\times H$ has vertex set $V\times W$

edges $\{((a,b),(\hat a,b)) : (a,\hat a)\in E\} \cup$
$\{((a,b),(a,\hat b)) : (b,\hat b)\in F\}$

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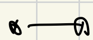
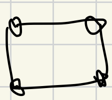
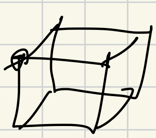
proof $(L\gamma_{i,j})(a,b) = \sum_{a,\hat a\in E}(\gamma_{i,j}(a,b)-\gamma_{i,j}(\hat a,b)) + \sum_{b,\hat b\in F}(\gamma_{i,j}(a,b)-\gamma_{i,j}(a,\hat b))$

$= \sum_{a,\hat a\in E}\beta_j(b)(\alpha_i(a)-\alpha_i(\hat a)) + \sum_{b,\hat b\in F}\alpha_i(a)(\beta_j(b)-\beta_j(\hat b))$

$= \beta_j(b)\cdot\lambda_i\alpha_i(a) + \alpha_i(a)\mu_j\beta_j(b) = (\lambda_i+\mu_j)\alpha_i(a)\beta_j(b) = (\lambda_i+\mu_j)\gamma_{ij}(a,b)$

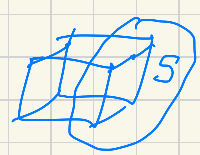
Ex. Hypercube $H_1 = P_2 =$ 
$H_d = P_2 \times H_{d-1}$





etc.

Eigs of L_{H_d} are $2i$ with multiplicity $\binom{d}{i}$ for $0\le i\le d$



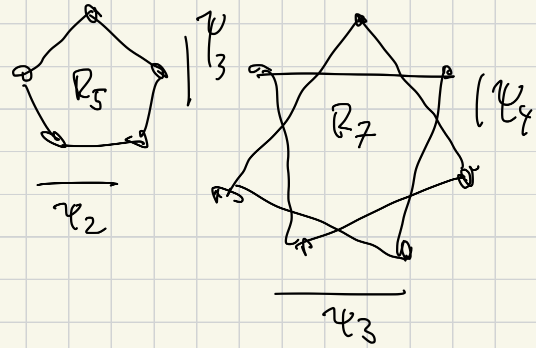
Can build eigvecs by inductive construction.

For $L_{H_{d-1}}\psi = \lambda\psi$ $L_{H_d}\begin{pmatrix}\psi\\ \psi\end{pmatrix} = \lambda\begin{pmatrix}\psi\\ \psi\end{pmatrix}$ $L_{H_d}\begin{pmatrix}\psi\\ -\psi\end{pmatrix} = (\lambda+2)\begin{pmatrix}\psi\\ -\psi\end{pmatrix}$

Usually index by $x\in\{0,1\}^d$. For vertex $a\in\{0,1\}^d$ $\psi_x(a) = (-1)^{a^Tx}$ $\lambda_x = 2\sum_i x(i)$

$\theta(S) = \frac{\partial(S)}{|S|} \ge \lambda_2\left(1-\frac{|S|}{n}\right)$. For $|S|=\frac{n}{2}$, get $\frac{\partial(S)}{|S|}\ge 1$. Is tight. Consider $S=\{a : a(1)=0\}$

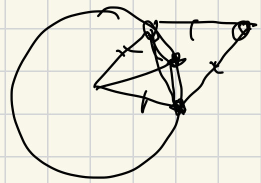
Ring R_n $V=\{1,\dots,n\}$ $E=(1,n)\cup\{(a,a+1):1\le a<n\}$



For $1\le k<\frac{n}{2}$ R_n has eigval $2-2\cos\left(\frac{2\pi k}{n}\right)$

with eigvecs $x_k(a)=\cos\left(\frac{2\pi k}{n}\right)$ $y_k(a)=\sin\left(\frac{2\pi k}{n}\right)$

Also, $x_0=\mathbb{1}$. if n even also is $x_{\frac{n}{2}}(a)=(-1)^a$



$$\cos\left(\frac{2\pi k a}{n}+\theta\right)=\cos\theta\cos\frac{2\pi k a}{n}-\sin\theta\sin\frac{2k\pi a}{n}$$

Algebra: $(L_{R_n}x_k)(a)=2x_k(a)-x_k(a+1)-x_k(a-1)$

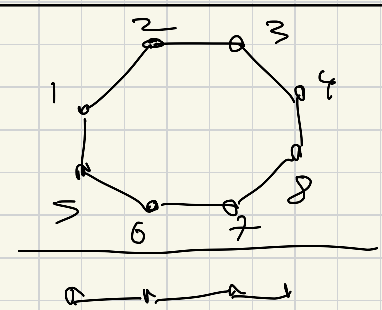
$$=2\cos\left(\frac{2\pi k a}{n}\right)-\cos\left(\frac{2\pi k a}{n}+\frac{2\pi k}{n}\right)-\cos\left(\frac{2\pi k a}{n}-\frac{2\pi k}{n}\right)$$

$$=2\cos\left(\frac{2\pi k a}{n}\right)-\cos\left(\frac{2\pi k a}{n}\right)\cos\left(\frac{2\pi k}{n}\right)+\sin\left(\frac{2\pi k a}{n}\right)\sin\left(\frac{2\pi k}{n}\right)-\cos\left(\frac{2\pi k a}{n}\right)\cos\left(\frac{2\pi k}{n}\right)-\sin$$

$$=2\cos\left(2\frac{\pi k a}{n}\right)\left(1-2\cos\left(\frac{2\pi k}{n}\right)\right)=x_k(a)\left(1-2\cos\left(\frac{2\pi k}{n}\right)\right)$$

<u>Thm</u> Path P_n has same eigvals as R_{2n}, excluding 2

<u>Proof</u> Rotate ring so it looks like



Under correct ordering $\begin{pmatrix}I_n\\I_n\end{pmatrix}^T L_{R_{2n}}\begin{pmatrix}I_n\\I_n\end{pmatrix}=2L_{P_n}$

If ψ is eigvec of R_{2n} s.t. $\psi(a)=\psi(a+n)$, $\phi(a)=\psi(a)$ $a=1..n$ has same eigval

$\psi=\begin{bmatrix}\phi\\\phi\end{bmatrix}$ so $2L_{P_n}\phi=\begin{pmatrix}I_n\\I_n\end{pmatrix}^T L_{R_{2n}}\begin{pmatrix}I_n\\I_n\end{pmatrix}\phi=\begin{pmatrix}I_n\\I_n\end{pmatrix}^T L_{R_{2n}}\psi=\begin{pmatrix}I_n\\I_n\end{pmatrix}\lambda\psi=2\lambda\phi$