Fiedler Let G be a connected graph, $L_G \psi_k = \lambda_k \psi_k$, and $W = \{a : \psi_k(a) \geq 0\}$

induced subgraph

Then $G(W)$ has at most $k-1$ connected components. (if $k=2$, $G(W)$ is connected).

$\Leftrightarrow$ If $L\psi = \lambda\psi$, $W = \{a : \psi(a) \geq 0\}$, then $\#\{\text{eigvals} < \lambda\} \geq \#$ of components of $G(W)$

Thm Let A be a matrix with non-pos off diagonals, like a Laplacian or submatrix of one, such that graph of non-zero off diagonals is connected, with eigvals $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. Then $\lambda_1 < \lambda_2$ and λ_1 has a strictly positive eigvec.

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proof Consider $\sigma I - A$. For big σ, is non-negative. Same eigvecs as A.

Eigvals $\sigma - \lambda_i$. So, PF $\Rightarrow$ $\sigma - \lambda_1 > \sigma - \lambda_2 \Rightarrow \lambda_1 < \lambda_2$, and positive eigvec.

proof of Fiedler Assume $G(W)$ has h components. Order vertices so all those come first, and $\{a : \psi(a) < 0\}$ so last. Write

$$L = \begin{pmatrix} B_1 & & 0 & C_1 \\ & B_2 & & \vdots \\ 0 & & \ddots \; B_h & C_h \\ C_1^T & \cdots & C_h^T & D \end{pmatrix} \qquad \psi = \begin{pmatrix} x_1 \\ \vdots \\ x_h \\ y \end{pmatrix}$$

$x_i \geq 0 \quad y \leq 0$

Graph of non-zeros in B_i is connected.

$C_i \leq 0$, not identically 0 because G connected

Will show $\lambda_{min}(B_i) < \lambda$. $\begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_h \end{pmatrix}$ has $\lambda_h < \lambda \Rightarrow \lambda_h(L) < \lambda$ by Cauchy Interlacing

$B_i x_i + C_i y_i = \lambda x_i \qquad C_i \leq 0,\ C_i \neq \bar{0},\ y_i < 0 \Rightarrow C_i y_i \geq 0,\ \neq \bar{0}.\ x_i \neq \bar{0}$

$x_i^T B_i x_i + x_i^T C_i y_i = \lambda x_i^T x_i \quad$ as $\quad x_i \geq 0,\ x_i^T C_i y_i \geq 0 \Rightarrow x_i^T B_i x_i \leq \lambda x_i^T x_i$

If $x_i > 0$, $x_i^T C_i y_i > 0$, and $x_i^T B_i x_i < \lambda x_i^T x_i \Rightarrow \lambda_{min}(B_i) < \lambda$

If $\exists a$ s.t. $x_i(a) = 0$, PF $\Rightarrow$ x_i not eigvec of $\lambda_{min}(B_i) \Rightarrow \lambda_{min}(B_i) < \lambda$

Partitioning & Clustering: remove many vertices by cutting few edges

Boundary of $S \subseteq V$ is $\partial(S) = \{(a,b) \in E : a \in S, b \notin S\}$

Isoperimetric ratio $\theta(S) = \dfrac{|\partial(S)|}{|S|}$ want S with $\theta(S)$ small, $|S| \le \frac{n}{2}$

$\theta_G = \min_{|S| \le \frac{n}{2}} \theta(S)$. Will show $\theta_G \ge \lambda_2/2$ $(*)$

later, for d-regular G $\theta_G \le \sqrt{2d\lambda_2}$

<u>Thm</u> $\theta(S) \ge \lambda_2(1-\sigma)$ where $\sigma = \frac{|S|}{n}$ (implies $(*)$)

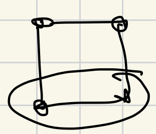
<u>proof</u> $\lambda_2 = \min_{x \perp \mathbb{1}} \dfrac{x^T L x}{x^T x}$ Try $x = \mathbb{1}_S - \sigma \mathbb{1}$, note $x^T \mathbb{1} = 0$ so

$\lambda_2 \le \dfrac{x^T L x}{x^T x}$ $x^T x = |S|(1-\sigma)$

As $L\mathbb{1} = 0$ and $\mathbb{1}^T L = 0^T$, $x^T L x = \mathbb{1}_S^T L \mathbb{1}_S = \sum_{(a,b) \in E} (\mathbb{1}_S(a) - \mathbb{1}_S(b))^2 = |\partial(S)|$

So, $\lambda_2 \le \dfrac{|\partial(S)|}{|S|(1-\sigma)} = \dfrac{\theta(S)}{1-\sigma}$

<u>Example</u>



$\lambda_2 = 2$

$|\partial(S)| = 2$ $|S| = 2$ $\theta(S) = 1$ $\sigma = \frac{1}{2}$ $\lambda_2(1-\sigma) = 1$ tight

Conductance: count vertices by degree. Allow weighted edges

$\phi(S) = \dfrac{w(\partial(S))}{\min(d(S), d(V-S))}$ $d(S) = \sum_{a \in S} d(a)$ $w(\partial(S)) = \sum_{(a,b) \in \partial(S)} w_{a,b}$

Relate to $\dfrac{y^T L y}{y^T D y}$, $\dfrac{\mathbb{1}_S^T L \mathbb{1}_S}{\mathbb{1}_S^T D \mathbb{1}_S} = \dfrac{w(\partial(S))}{d(S)}$

Set $x = D^{1/2} y$, $y = D^{-1/2} x$ so $\dfrac{y^T L y}{y^T D y} = \dfrac{x^T D^{-1/2} L D^{-1/2} x}{x^T x}$ $N = D^{-1/2} L D^{-1/2}$ = normalized Lap

eigs $0 = \nu_1 \le \nu_2 \le \cdots \le \nu_n$

Let $d^{1/2}$ be vec s.t. $d^{1/2}(a) = \sqrt{d(a)}$. Is eigvec of 0. $N d^{1/2} = D^{-1/2} L D^{-1/2} d^{1/2} = D^{-1/2} L \mathbb{1} = 0$

$\nu_2 = \min_{x : x^T d^{1/2} = 0} \dfrac{x^T N x}{x^T x}$ $x^T d^{1/2} = 0 \leftrightarrow y^T d = 0$ so $\nu_2 = \min_{y : y^T d = 0} \dfrac{y^T L y}{y^T D y}$

Can prove $\forall S$ $\phi(S) \geq \nu_2(1-\sigma)$ where $\sigma = \frac{d(S)}{d(V)}$

Like to state as $\frac{w(\partial(S))}{d(S)\,d(V-S)} \cdot d(V) \geq \nu_2$

Define $\phi_G = \min_S \phi(S)$. So $\phi_G \geq \nu_2/2$

Will show $\phi_G \leq \sqrt{2\nu_2}$ $\Rightarrow$ $\exists S$ s.t. $\phi(S) \leq \sqrt{2\nu_2}$ - Cheeger's inequality.