Lect 4. Coloring & Independent Sets (1/24/25)

1. Finish PF, bipartite graphs
2. Coloring & Indep sets
3. Indep set upper bound
4. Interlacing
5. Chromatic lower bound.

For M = adj of connected graph + pos diagonal

a. μ_1 has a strictly pos eigvec

b. $\mu_1 > \mu_2$

c. $\mu_1 \geq |\mu_n|$

proof (c) let $\psi_n \in$ unit eigvec of μ_n. $y(a) = |\psi_n(a)|\ \forall a$

$$|\mu_n| = |\psi_n^T M \psi_n| = \left|\sum_{a,b} M(a,b)\psi_n(a)\psi_n(b)\right| \leq \sum_{a,b} M(a,b)\, y(a)\, y(b) = y^T M y \leq \mu_1 \quad (*)$$

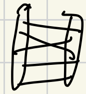
Thm 1. $\mu_1 = \mu_n \Rightarrow G$ is bipartite ($V = (U, W)$ all edges from U to W)

proof All edges (a,b) connect pos to neg in ψ_n ($U = \{a : \psi(a) > 0\}$, $W = \{a : \psi(a) < 0\}$)

$(*)$ tight $\Rightarrow y$ an eigvec of $\mu_1 \Rightarrow$ strictly pos $\Rightarrow \psi_n$ never zero

$\leq$ in $(*)$ tight $\Rightarrow$ all terms same sign $\Rightarrow \psi_n(a)\psi_n(b) < 0$

$\forall (a,b) \in E$

Thm 2 Bipartite $\Rightarrow$ μ eigval $\Leftrightarrow$ $-\mu$ eigval 

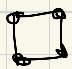
G bipartite $\Rightarrow$ can order vertices so $M = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$ some B

If $Mx = \mu x$ write as $\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \mu \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} Bx_1 \\ B^T x_0 \end{pmatrix}$

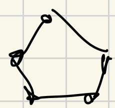
So $\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} -x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} Bx_1 \\ -B^T x_0 \end{pmatrix} = \begin{pmatrix} \mu x_0 \\ -\mu x_1 \end{pmatrix} = -\mu \begin{pmatrix} -x_0 \\ x_1 \end{pmatrix}$

A k-coloring of a graph $G = (V, E)$ is

$c : V \to \{1, .., k\}$ s.t. $c(a) \neq c(b)$ $\forall (a,b) \in E$



2-colorable



not 2-colorable

2-colorable = bipartite

Planar graphs are 4-colorable

Chromatic #. $\chi(G) = \min\{k : k\text{-colorable}\}$

$S \subseteq V$ is independent if $G(S)$ has no edges.

$\alpha(G) = \max\{|S| : S \text{ is independent}\}$

High freq eigvecs (λ_n, μ_n) push neighbors apart.

Try to color.

Hoffman: If G d-regular, S indep, then $|S| \le n\left(\frac{-\mu_n}{d-\mu_n}\right)$

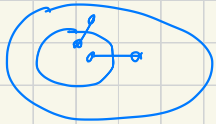
(Bipartite $\Rightarrow \mu_n = -d$, $|S| \le \frac{n}{2}$.)

$$= n\left(\frac{\lambda_n - d}{\lambda_n}\right) = n\left(1-\frac{d}{\lambda_n}\right)$$

<u>Thm 3</u> For $S \subseteq V$, let $d_{ave}(S) = \frac{1}{|S|}\sum_{a\in S} d(a)$ (not $d_{ave}(G(S))$)

If S is independent, $|S| \le n\left(1 - \frac{d_{ave}(S)}{\lambda_n}\right)$

<u>proof</u> $\lambda_n = \max_x \frac{x^T L x}{x^T x}$



Consider $x = \mathbb{1}_S - s\mathbb{1}$, so $\mathbb{1}^T x = |S| - sn = 0$

$$x^T L x = \mathbb{1}_S^T L \mathbb{1}_S^T = \sum_{(a,b)\in E} (\mathbb{1}_S(a) - \mathbb{1}_S(b))^2 = d(S) = |S|\, d_{ave}(S)$$

$$x^T x = |S|(1-s)^2 + (n-|S|)s^2 = ns(1-s) = |S|(1-s)$$

So, $\lambda_n \ge \frac{|S|\, d_{ave}(S)}{|S|(1-s)} = \frac{d_{ave}(S)}{(1-s)} \iff 1-s \ge \frac{d_{ave}(S)}{\lambda_n}$

$$\iff s \le 1 - \frac{d_{ave}(S)}{\lambda_n}$$

If k-colorable, is an indep set of size $\ge \frac{n}{k}$

So, $\frac{n}{k} \le n\left(\frac{-\mu_n}{d-\mu_n}\right) \iff k \ge \frac{d-\mu_n}{-\mu_n} = 1 + \frac{d}{-\mu_n}$

Will show $k \ge 1 + \frac{\mu_1}{-\mu_n}$

<u>Cauchy's Interlacing Theorem</u> If A is symmetric, eigvals $\alpha_1 \ge \cdots \ge \alpha_n$

B a principal submatrix (remove 1 row & col), eigs $\beta_1 \ge \cdots \ge \beta_{n-1}$

Then $\alpha_1 \ge \beta_1 \ge \alpha_2 \ge \beta_2 \ge \cdots \ge \beta_{n-1} \ge \alpha_n$

Proof wolog, assume remove first row & col

let's show $\alpha_k \geq \beta_k$

CF: $\beta_k = \max\limits_{\dim(S)=k} \min\limits_{x \in S} \dfrac{x^T B x}{x^T x} = \max\limits_{\dim(S)=k} \min\limits_{x \in S} \dfrac{\begin{pmatrix}0\\x\end{pmatrix}^T A \begin{pmatrix}0\\x\end{pmatrix}}{\begin{pmatrix}0\\x\end{pmatrix}^T \begin{pmatrix}0\\x\end{pmatrix}}$

$$\leq \max_{\dim(S)=k} \min_{x \in S} \frac{x^T A x}{x^T x} = \alpha_k$$

$\alpha_{k+1} \leq \beta_k$ follows from applying to $-A$ and $-B$, $k \to n-k$

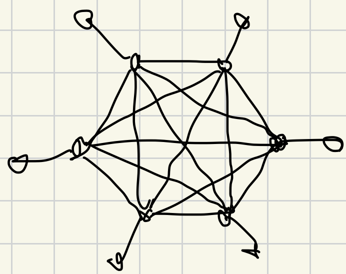
Thm 4 $\chi \geq 1 + \dfrac{\mu_1}{-\mu_n}$

$\chi = 6$

$\mu_1 = 5.19$

$\mu_n = -1.62$

$1 + \dfrac{\mu_1}{-\mu_n} = 4.2$



Doubling clique weights → 5.18

Lem 5 If $M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,k} \\ M_{12}^T & M_{22} & & \\ & & \ddots & \\ & & & M_{k,k} \end{pmatrix}$

M symmetric

$$\lambda_{max}(M) \leq \sum_i \lambda_{max}(M_{i,i}) - (k-1)\lambda_{min}(M)$$

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proof of Thm 4

k-colorable $\Rightarrow$ can order vertices so that $M_{i,i} = 0 \ \forall i$

$$\Rightarrow \mu_1 \leq -(k-1)\mu_n = (k-1)(-\mu_n) \iff k \geq 1 + \frac{\mu_1}{-\mu_n}$$

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proof of lem 5 $\quad k=2, \quad M=\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$

$$\lambda_{max}(M) + \lambda_{min}(M) \le \lambda_{max}(A) + \lambda_{max}(C)$$

Let $\begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$ be unit eigvec of eigval μ_1

If $x_1 = 0$, $\quad x^T M x = x^T A x \to \lambda_{max}(A) \ge \lambda_{max}(M)$

and, $\lambda_{max}(C) \ge \lambda_{min}(C) \ge \lambda_{min}(M)$ by Cauchy

O/w, set $y = \begin{pmatrix} \frac{\|x_1\|}{\|x_0\|} x_0 \\ -\frac{\|x_0\|}{\|x_1\|} x_1 \end{pmatrix} \qquad \|y\| = \|x\| = 1$

$$x^T M x + y^T M y \ge \lambda_{max}(M) + \lambda_{min}(M)$$

$$x^T M x + y^T M y \le \left(1 + \frac{\|x_1\|^2}{\|x_0\|^2}\right) x_0^T A x_0 + \left(1 + \frac{\|x_0\|^2}{\|x_1\|^2}\right) x_1^T C x_1$$

$$\le \left(\|x_1\|^2 + \|x_0\|^2\right)\left(\lambda_{max}(A) + \lambda_{max}(C)\right)$$

Induction $\quad A = \begin{pmatrix} M_{11} & & \\ & \ddots & \\ & & M_{k-1,k-1} \end{pmatrix} \qquad C = M_{k,k}$

$$\lambda_{max}(M) \le \lambda_{max}(A) + \lambda_{max}(C) - \lambda_{min}(M)$$

$$\leq \sum_{i=1}^{k-1} \lambda_{max}(M_{i,i}) - (k-2)\lambda_{min}(A) + \lambda_{max}(M_{k,k}) - \lambda_{min}(M)$$

$$\lambda_{min}(A) \geq \lambda_{min}(M)$$

$$\leq \sum_{i=1}^{k} \lambda_{max}(M_{i,i}) - (k-1)\lambda_{min}(M)$$

Wilf's coloring $\chi \leq 1 + \mu_1$?