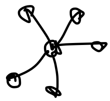
Adjacency Matrix Eigenvalues, Jan 22, 2025

To start, compute adjacency eigvals of star graph



$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & & & & & \\ \vdots & & & 0 & & \\ 1 & & & & & \end{pmatrix} = v\delta_1^T + \delta_1 v^T \text{ where } v(a) = \begin{cases} 0 & a=1 \\ 1 & \text{o.w.} \end{cases}$$

i. M has rank 2, so $n-2$ eigvals are 0.

ii. $Tr(M) = \sum_i \mu_i = 0$. So, non-zero eigvals are $\mu_1 > 0 > \mu_n$

$$\mu_n = -\mu_1$$

iii. Consider M^2. $\text{eigvals}(M^2) = (\text{eigvals}(M))^2$

$$M^2 = \begin{pmatrix} n-1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \qquad Tr(M^2) = 2(n-1) = \mu_1^2 + \mu_n^2$$

$$\mu_1 = \sqrt{n-1} \qquad \mu_n = \sqrt{n-1}$$

Eigvals are $\pm\sqrt{n-1}$, 0^{n-2}

Last lecture saw Laplacian eigenvalues $0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$

$\lambda_1 = \lambda_2$ iff disconnected.

Today, look at Adjacency eigenvalues $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n$

If d-regular, $\mu_i = d - \lambda_i$, $\mu_1 = d$

But, interested in irregular case.

Will show $\mu_1 = \mu_2$ iff disconnected, $\mu_n = -\mu_1$ iff bipartite,

chromatic # $\le 1 + \lfloor \mu_1 \rfloor$? probably next lecture

<u>Thm 1</u> For a weighted graph, $d_{ave} \le \mu_1 \le d_{max}$, d_{max} = max wted degree

<u>proof</u> $d_{ave} = \frac{1}{n} \sum_a d(a) = \frac{\mathbb{1}^T d}{n} = \frac{\mathbb{1}^T M \mathbb{1}}{\mathbb{1}^T \mathbb{1}} \le \mu_1$ by Courant-Fischer

Let ψ_1 be eigvec of μ_1, $a \in \arg\max_a \psi_1(a)$. wlog $\psi_1(a) > 0$.

$$\mu_1 \cdot \psi_1(a) = (M\psi_1)(a) = \sum_{b \sim a} w_{a,b}\, \psi_1(b) \le \sum_{b \sim a} w_{a,b}\, \psi_1(a) = \psi_1(a)\, d(a)$$

$$\le \psi_1(a)\, d_{max}(a) \quad (*)$$

Is this a good bound? Answer 1:

<u>Thm 2</u> If $\mu_1 = d_{max}$ and G connected, then $d_{ave} = d_{max}$

That is, G is μ_1-regular.

proof: If have equality in $(*)$,

then $d(a) = d_{max}$ and $\psi_1(b) = \psi_1(a)$ $\forall b \sim a$

So, $b \in \arg\max_a \psi_1(a)$, and can apply this arg to every neighbor of a. As G is connected, eventually reach every vertex and conclude $\psi_1 = \psi_1(a) \cdot \mathbb{1}$, $d(b) = d_{max}$ $\forall b \in V$.

Answer 2: μ_1 does a lot of work

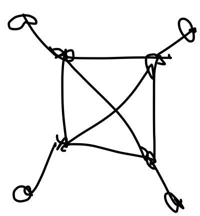
For $S \subseteq V$, let $G(S)$ be subgraph induced on S.

Vertex set is S, edges $\{(a,b) \in E : a \in S, b \in S\}$

prop 3 $d_{ave}(G(S)) \leq \mu_1$

proof. Consider $\mathbb{1}_S$. $\mu_1 \geq \dfrac{\mathbb{1}_S^T M \mathbb{1}_S}{\mathbb{1}_S^T \mathbb{1}_S} = \dfrac{\sum_{\substack{a,b \in S \\ a \sim b}} w_{a,b}}{|S|} = d_{ave}(G(S))$

And, we can have $d_{ave}(G(S)) > d_{ave}(G)$



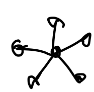
$d_{ave} = 2\frac{1}{2}$

$\mu_1 = 3.303\ldots$

But $\mu_1(G(S)) \leq \mu_1(G)$, always

(Next class)

Answer 3: can be a poor bound on μ_1

Ex: star on n vertices  $n=6$, $d_{ave} = \frac{2(n-1)}{n}$ $\mu_1 = \sqrt{n-1}$

If goal is an upper bound on μ_1, can do better

Def For an arbitrary matrix M, $\|M\|_\infty \overset{\text{def}}{=} \max_a \sum_b |M(a,b)|$

Thm 4 $M\psi = \mu\psi \Rightarrow |\mu| \le \|M\|_\infty$

proof let $a \in \arg\max_a |\psi(a)|$. Then

$$|\mu| \cdot |\psi(a)| = |\mu\psi(a)| = |(M\psi)(a)| = \left|\sum_b M(a,b)\,\psi(b)\right| \le |\psi(b)| \left(\sum_b |M(a,b)|\right)$$

$$\le |\psi(a)|\,\|M\|_\infty$$

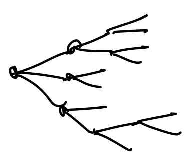
To improve bound for star, let $M = \begin{pmatrix} 0 & 1 \cdots 1 \\ 1 & \\ \vdots & 0 \\ 1 & \end{pmatrix}$, $d = n-1$

let $D = \begin{pmatrix} \sqrt{d} & & \\ & 1 & \\ & & 1 \end{pmatrix}$, and consider $D^{-1}MD = \begin{pmatrix} 0 & 1/\sqrt{d} \cdots 1/\sqrt{d} \\ \sqrt{d} & \\ \vdots & 0 \\ \sqrt{d} & \end{pmatrix}$

$D^{-1}MD$ has same eigvals as M, $\left(M\psi = \mu\psi \Leftrightarrow (D^{-1}MD)(D^{-1}\psi) = \mu(D^{-1}\psi) \right)$

But $\|D^{-1}MD\|_\infty = \sqrt{d} \ge \mu_1$, which is tight.

In the book, we do the same for all d-ary trees:



we prove $\mu_1 \le 2\sqrt{d-1}$. is tight.

$2\sqrt{d-1}$ comes up a lot

Symmetric Perron-Frobenius Theory

For a connected, weighted graph

a. μ_1 has a strictly positive eigenvector

b. $\mu_1 > \mu_2$

c. $\mu_1 \geq |\mu_n|$

<u>lem 5</u> let $M\psi = \mu\psi$, $\psi(a) \geq 0 \ \forall a$, $\psi \neq \bar{0}$.

Then $\psi(a) > 0 \quad \forall a$

<u>proof</u> Assume b.w.o.c. $\exists a$ s.t. $\psi(a) = 0$

G connected $\Rightarrow \exists (b,c) \in E$ s.t. $\psi(b) = 0 < \psi(c)$

$$0 = \mu\psi(b) = (M\psi)(b) = \sum_{z\sim b} \underbrace{M(b,z)}_{\geq 0}\ \underbrace{\psi(z)}_{\geq 0} \geq M(b,c)\,\psi(c) > 0$$

<u>proof of PF</u>

a. Let $\psi_1 \in$ unit eigvecs of μ_1. let $x(a) = |\psi_1(a)|$, $\forall a$

Will show $x \in$ unit eigvecs of μ_1. lem 5 $\rightarrow$ $x(a) > 0 \ \forall a$

$$\mu_1 = \psi_1^T M \psi_1 = \sum_{a,b} M(a,b)\,\psi_1(a)\,\psi_1(b) \leq \sum_{a,b} M(a,b)\,|\psi_1(a)|\,|\psi_1(b)|$$

$$= x^T M x \leq \mu_1. \quad \text{Maximum}, \ x^T x = \psi_1^T \psi_1 \Rightarrow x \in \text{eigvecs of } \mu_1$$

b. $\mu_2 < \mu_1$. Let ψ_2 be a unit eigvec of μ_2.

As $\psi_2^T \psi_1 = 0$, ψ_2 has pos and neg entries.

Let $y(a) = |\psi_2(a)|$, $\forall a$.

$\mu_2 = \psi_2^T M \psi_2 \leq y^T M y \leq \mu_1$

Assume, bwoc, $\mu_2 = \mu_1$. So, y is non-neg eigvec of μ_1

$\xrightarrow{\text{Lem 5}}$ y strictly pos $\rightarrow$ ψ_2 never zero

Connected $\rightarrow$ $\exists (a,b) \in E$ s.t. $\psi_2(a) < 0 < \psi_2(b)$

$\rightarrow$ $\psi_2 M \psi_2 < y^T M y$, because $M(a,b)\psi_2(a)\psi_2(b) < 0 < M(a,b) y(a) y(b)$

$\#$

c. Let $\psi_n \in$ unit eigvec of μ_n. $y(a) = |\psi_n(a)|$, $\forall a$

$|\mu_n| = |\psi_n^T M \psi_n| \leq y^T M y \leq \mu_1 \qquad (*)$

Thm If G connected, $\mu_n = -\mu_1$ iff G is bipartite

proof $\Rightarrow$ $(*)$ tight $\Rightarrow$ y is pos eigvec of μ_1, $\psi_n(a) > 0$, $\forall a$

$$\Rightarrow \Big| \underbrace{\sum_{a,b} M(a,b)\psi_n(a)\psi_n(b)}_{\text{all these terms have same sign.}} \Big| = \sum_{a,b} M(a,b) |\psi_n(a)| \cdot |\psi_n(b)|$$

As in part b, $\exists (a,b) \in E$ s.t. $\psi_n(a) < 0 < \psi_n(b)$, so sign is $-$

$\Rightarrow$ for all $(a,b) \in E$, $\psi_n(a)\psi_n(b) < 0$.

Bipartition into pos and neg entries in ψ_n

$\Leftarrow$ Will show for Bipartite μ eigval $\Leftrightarrow$ $-\mu$ eigval

G bipartite $\Rightarrow$ Can order vertices so that

$$M = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \quad \text{for some } B$$

Let $M\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \mu \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$ so $\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} Bx_1 \\ B^T x_0 \end{pmatrix} = \begin{pmatrix} \mu x_0 \\ \mu x_1 \end{pmatrix}$

Then $M\begin{pmatrix} x_0 \\ -x_1 \end{pmatrix} = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ -x_1 \end{pmatrix} = \begin{pmatrix} -Bx_1 \\ B^T x_0 \end{pmatrix} = \begin{pmatrix} -\mu x_0 \\ \mu x_1 \end{pmatrix} = -\mu \begin{pmatrix} x_0 \\ -x_1 \end{pmatrix}$

So, $-\mu$ is an eigenvalue.

Next 2 lectures: make qualitative quantitative.

Bipartite $\Leftrightarrow$ 2-colorable.

chromatic number $\geq 1 + \dfrac{\mu_1}{-\mu_n}$ (Hoffman)

Connected $\Leftrightarrow \mu_2 < \mu_1 \quad \rightarrow \quad \mu_2 \ll \mu_1 \Leftrightarrow$ very connected

$\lambda_2 \gg 0$ (Fiedler / Cheeger)

$\mu_1 \gg |\mu_n| \Leftrightarrow$ far from bipartite (Trevisan)