1. Courant – Fischer

2. The Laplacian

3. Hall's Graph Drawing, using Laplacian

Thm! Let M be a symmetric matrix and let x maximize

$$\frac{x^T M x}{x^T x}. \quad \text{Then } Mx = \mu x \text{ where } \mu = \frac{x^T M x}{x^T x}$$

proof First, claim x exists.

Why? Analysis

Can assume $\|x\| = 1$. Set of unit vectors is closed & compact
function is continuous. So, max is achieved.

At max, gradient $= 0$.

$$\nabla x^T x = 2x \qquad \nabla x^T M x = 2Mx$$

$$\nabla \frac{x^T M x}{x^T x} = \frac{(x^T x)(2Mx) - (x^T M x)(2x)}{(x^T x)^2} = 0$$

$$\Rightarrow (x^T x)(2Mx) = (x^T M x)(2x)$$

$$\Rightarrow Mx = \left(\frac{x^T M x}{x^T x}\right) x$$

Can use this to prove the spectral Theorem

Spectral Thm An n-by-n symmetric matrix M has

n real eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$

and orthonormal eigenvectors $\psi_1, \dots, \psi_n$ s.t. $M = \sum_i \mu_i \psi_i \psi_i^T$

Idea: Use Thm 1 to find ψ_1, μ_1, $\|\psi_1\| = 1$

Set $M_1 = M - \mu_1 \psi_1 \psi_1^T$

now, $M_1 \psi_1 = M\psi_1 - \mu_1 \psi_1 (\psi_1^T \psi_1) = 0$. lower rank

Apply Thm 1 to M_1 to get ψ_2, etc.

Is in book. Section 2.2.

Courant - Fischer Theorem.

$$\mu_k \overset{(1)}{=} \max_{\dim(S)=k} \min_{x \in S} \frac{x^T M x}{x^T x} = \min_{\dim(S)=n-k+1} \max_{x \in S} \frac{x^T M x}{x^T x}$$

look at $k=1$

$$\psi_k \in \arg\max_{\substack{\|x\|=1 \\ x \perp \psi_1, \dots, \psi_{k-1}}} x^T M x \qquad \psi_k \in \arg\min_{\substack{\|x\|=1 \\ x \perp \psi_{k+1}, \dots, \psi_n}} x^T M x$$

proof of (1) $\geq$ let $S = \text{span}(\psi_1, \dots, \psi_k)$

For $x \in S$, write $x = \sum_{i=1}^{k} c_i \psi_i$

$$\frac{x^T M x}{x^T x} = \frac{\sum_{i=1}^{k} c_i^2 \mu_i}{\sum_{i=1}^{k} c_i^2} \geq \frac{\mu_k \sum_{i=1}^{k} c_i^2}{\sum_{i=1}^{k} c_i^2} = \mu_k$$

by this arg

$\leq$ let S have dim k.

let $T = \mathrm{span}(\psi_k, \ldots, \psi_n)$, has dim $n-k+1$

so $S \cap T$ is non-empty

$$\min_{x \in S} \frac{x^T M x}{x^T x} \leq \min_{x \in S \cap T} \frac{x^T M x}{x^T x} \leq \max_{x \in T} \frac{x^T M x}{x^T x} \leq \mu_k$$

<u>Laplacians</u> Want a matrix L s.t. $x^T L x = \sum_{(a,b) \in E} w_{a,b} (x(a) - x(b))^2$

Consider $G_{a,b}$ = graph with just edge (a,b)

$$x^T L_{G_{a,b}} x = (x(a) - x(b))^2 = ((\delta_a - \delta_b)^T x)^2$$

$$= x^T (\delta_a - \delta_b)(\delta_a - \delta_b)^T x$$

$$(\delta_1 - \delta_2)(\delta_1 - \delta_2)^T = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Summing gives

$$L = \sum_{(a,b) \in E} w_{a,b} (\delta_a - \delta_b)(\delta_a - \delta_b)^T$$

$$= D - M$$

$$(Lx)(a) = d_a x(a) - \sum_{b \sim a} w_{a,b}\, x(b) = \sum_{b\sim a} w_{a,b}\,(x(a) - x(b))$$

Basic Facts:

$L\mathbb{1} = 0$, from $\mathbb{1}^T L \mathbb{1}$ or $M\mathbb{1} = d$ $D\mathbb{1} = d$

$x^T L x \geq 0 \ \forall x.$ $\lambda_1 = 0$

NOTE: order Laplacian eigvals $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$

<u>Thm</u> $\lambda_2 = 0$ iff G is disconnected

<u>proof</u> If G is disconnected, and $S, T \subseteq V$ are components

Consider vectors $\mathbb{1}_S$ and $\mathbb{1}_T$

$$\mathbb{1}_S(a) = \begin{cases} 1 & a \in S \\ 0 & \text{o.w.} \end{cases}$$

$L\mathbb{1}_S = 0$, because all nbrs of S are in S

$L\mathbb{1}_T = 0$

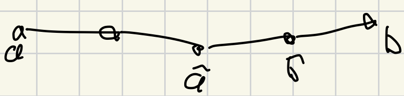
$\mathbb{1}_S^T \mathbb{1}_T = 0$

So, are two eigvals of 0.

Conversely, if G is connected, let x be any

non-constant vector. Claim $x^T L x > 0$.

$\exists\, a \neq b$ s.t. $x(a) \neq x(b)$



is a path from a to b. So are $\hat{a}, \hat{b}$ on path

s.t. $\hat{a}, \hat{b} \in E$, $x(\hat{a}) \neq x(\hat{b})$, $(x(\hat{a}) - x(\hat{b}))^2 > 0$

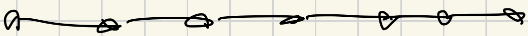
Other terms non-negative $\Rightarrow$ $x^T L x > 0$

Hall's Spectral Graph Drawing

Think of what want in a drawing,

phrase as constrained optimization.

For example, if have path, want to draw it like



Drawing on line / ordering vertices want edges to be short.

Consider $\min x^T L x$

sol could be zero – so require $\|x\| = 1$

sol could be $\frac{1}{\sqrt{n}}\mathbb{1}$ – so require $\sum_a x(a) = 0 = \mathbb{1}^T x$

Get $\psi_2 \in \arg\min_{\substack{\|x\|=0 \\ x \perp \mathbb{1}}} x^T L x$

In 2D, map a to $(x(a), y(a))$

$$\min \sum_{(a,b)\in E} \left\| \begin{pmatrix} x(a) \\ y(a) \end{pmatrix} - \begin{pmatrix} x(b) \\ y(b) \end{pmatrix} \right\|^2$$

$$= \sum_{(a,b)\in E} (x(a)-y(a))^2 + (x(b)-y(b))^2 = x^T L x + y^T L y$$

s.t. $\|x\|=1$

$\|y\|=1$

$x^T \mathbb{1} = 0$

$y^T \mathbb{1} = 0$

want to avoid $x=y$, so add $x^T y = 0$

Would guess $x=\psi_2$ $y=\psi_3$, or any rotation.

Hall: to draw in $\mathbb{R}^k$

find $x_1, \ldots, x_k$ unit vecs, orth to $\mathbb{1}$ and each other

$$\min \sum_{i=1}^{k} x_i^T L x_i$$

<u>Thm</u> min val is $\lambda_2 + \cdots + \lambda_{k+1}$, and sol is $x_i = \psi_{i+1}$

<u>proof</u> will show $\lambda_2 + \cdots + \lambda_{k+1}$ is a lower bound

Choose $x_{k+1}, \ldots, x_n$ to be an orthonormal basis of space orth to $x_1 \ldots x_k$, so $x_1 \ldots x_n$ is an orthonormal basis. Compare to $\psi_1 \ldots \psi_n$

$\forall i$ $\sum_j (\psi_j^T x_i)^2 = 1$ and $\forall j$ $\sum_i (\psi_j^T x_i)^2 = 1$

$\psi_1 = \frac{1}{\sqrt{n}} \mathbb{1}$

For $i \le k$, $x_i^T L x_i = \sum_{j=2}^{n} \lambda_j (\psi_j^T x_i)^2$

$$= \sum_{j=2}^{n} (\lambda_j - \lambda_{k+1}) (\psi_j^T x_i)^2 + \lambda_{k+1} \qquad \lambda_j \ge \lambda_{k+1} \text{ for } j \ge k+1$$

$$\ge \sum_{j=2}^{k+1} (\lambda_j - \lambda_{k+1}) (\psi_j^T x_i)^2 + \lambda_{k+1}$$

So $$\sum_{i=1}^{k} x_i^T L x_i \ge k \cdot \lambda_{k+1} + \sum_{j=2}^{k+1} \underbrace{(\lambda_j - \lambda_{k+1})}_{\le 0} \underbrace{\sum_{i=1}^{k} (\psi_j^T x_i)^2}_{\le 1}$$

$$\ge k \lambda_{k+1} + \sum_{j=2}^{k+1} \lambda_j - \lambda_{k+1}$$

$$= \sum_{j=2}^{k+1} \lambda_j$$

Notes: Good story, but did cause can compute

Is v_1 for graph drawing

Many other optimizations, like norm instead of norm^2

Repulsion to prevent overlap, etc.

Usually cannot minimize predictably

Get a picture, but not reality.

Rand perm vertices?