2025-Apr-2 For every weighted graph $G=(V,E,w)$ and $\varepsilon>0$ $\exists$ $H=(V,F,u)$ s.t.

$$(1-\varepsilon)G \preccurlyeq H \preccurlyeq (1+\varepsilon)G, \quad |F| \le \frac{4n\ln n}{\varepsilon^2} \text{ where } n=|V|, \text{ and } F\subseteq E.$$

G and H are similar in

sizes of boundaries of subsets

eigenvalues

inverses

effective resistances

Recall, $A \preccurlyeq B$ iff $\forall x$ $x^TAx \le x^TBx$

<u>Lem 1</u> For non-singular C

$$0 \preccurlyeq A \preccurlyeq B \Leftrightarrow C^TAC \preccurlyeq C^TBC$$

<u>Cor</u> $A \preccurlyeq B$ iff $B^{-1} \preccurlyeq A^{-1}$

$$A \preccurlyeq B \Leftrightarrow B^{-1/2}AB^{-1/2} \preccurlyeq I \Leftrightarrow B^{1/2}A^{-1}B^{1/2} \succcurlyeq I \Leftrightarrow A^{-1} \succcurlyeq B^{-1}$$

For every edge $e\in E$, choose $p_e \in [0,1]$.

Include e in F with prob p_e. If include, set $u_e = w_e/p_e$. Or, set $u_e=0$ if do not include

$$\mathbb{E}u_e = p_e(w_e/p_e) = w_e$$

$p_e = \alpha\, w_{a,b}\, R_{\text{eff}}(a,b)$

So, $$\mathbb{E}L_H = \sum_{(a,b)\in E} (\mathbb{E}u_e)\, L_{a,b} = L_G$$

Worry is deviations.

<u>Thm</u> Matrix Chernoff Bounds (Tropp '12)

Let $X_1,\dots,X_m$ be random psd matrices with $\|X_i\| \le R$. $X=\sum X_i$,

μ_{min} = smallest eig of $\mathbb{E}X$. μ_{max} = largest

$$\Pr[\lambda_{min}(X) \le (1-\varepsilon)\mu_{min}] \le n e^{-\frac{\varepsilon^2\mu_{min}}{2R}}$$

$$\Pr[\lambda_{max}(X) \ge (1+\varepsilon)\mu_{max}] \le n e^{-\frac{\varepsilon^2\mu_{max}}{3R}}$$

Transformation

Let $L^{+/2} = (L^+)^{1/2}$. Square root of pseudo-inverse

$(1-\varepsilon) L_G \preceq L_H \preceq (1+\varepsilon) L_G$

$\Updownarrow$ (Lem 1, ish)

$(1-\varepsilon)\Pi \preceq L_G^{+/2} L_H L_G^{+/2} \preceq (1+\varepsilon)\Pi$

where $\Pi = L_G^{+/2} L_G L_G^{+/2} = I - \frac{1}{n} J$

$L_G^{+/2} L_H L_G^{+/2} = \sum X_{a,b}$ where

$$X_{a,b} = \begin{cases} \frac{w_{a,b}}{p_{a,b}} L_G^{+/2} L_{a,b} L_G^{+/2} & \text{with prob } p_{a,b} \\ 0 & \text{with prob } 1-p_{a,b} \end{cases}$$

$L_{a,b} = (\delta_a - \delta_b)(\delta_a - \delta_b)^T$, so $\| L_G^{+/2} L_{a,b} L_G^{+/2} \| =$ take min with 1.

$(\delta_a - \delta_b)^T L_G^+ (\delta_a - \delta_b) = R_{eff}(a,b)$. $p_{a,b} = \gamma w_{a,b} R_{eff}(a,b)$

gives $\|X_{a,b}\| \in \{1/\gamma, 0\}$. So $R = 1/\gamma$ in Thm 1.

By Thm 1, prob is not an ε-approx $\leq 2n e^{-\frac{\varepsilon^2}{3R}}$, small if $R < \varepsilon^2 / 3\ln(2n)$

Set $R = \varepsilon^2 / 3.5\ln(2n) = 1/\gamma$

$\gamma = 3.5\ln(2n)/\varepsilon^2$

What if $p_{a,b} > 1$? Just include those edges and sample on rest

Let $E_0 =$ edges $p_{a,b} > 1$, $E_1 =$ rest of edges.

$\mu_{max} < 1$

Want $\left[\lambda_{max}\left(\sum_{(a,b)\in E_1} X_{a,b}\right) \leq \mu_{max}\left(\mathbb{E} \sum_{a,b \in E_1} X_{a,b}\right) + \varepsilon\right]$ let $\hat{\varepsilon} = \varepsilon/\mu_{max}$, so Thm 2 gives

Prob fail $\leq n\exp\left(-\frac{\hat{\varepsilon}^2 \mu_{max}}{3R}\right) = n\exp\left(\frac{-\varepsilon^2}{3R\mu_{max}}\right) < n e^{-\varepsilon^2/3R}$

How many edges? $\mathbb{E}\#\text{edges} = \sum_{a,b} p_{a,b} \le \sum_{a,b} \gamma w_{a,b} \mathrm{Reff}_G(a,b)$ $\qquad \gamma = 3.5\ln(2n)/\varepsilon^2$

$\sum_{a,b} w_{a,b} \mathrm{Reff}_G(a,b) = n-1$. So, $\mathbb{E}\#\text{edges} \le \frac{3.5 n \ln n}{\varepsilon^2}$

<u>proof</u> $\sum_{a,b} w_{a,b} (\delta_a-\delta_b)^T L^+ (\delta_a-\delta_b) = \sum_{a,b} w_{a,b} \mathrm{Tr}\left(L^+ (\delta_a-\delta_b)(\delta_a-\delta_b)^T\right)$

$= \mathrm{Tr}\left(\sum_{a,b} w_{a,b} L^+ L_{a,b}\right) = \mathrm{Tr}\left(L^+ \sum_{a,b} w_{a,b} L_{a,b}\right) = \mathrm{Tr}(L^+L) = n-1.$

Can use Hoeffding to prove unlikely # edges $> \frac{4 n \ln n}{\varepsilon^2}$

Stories, if time. Can quickly approximate $\mathrm{Reff}_G(a,b)$ $\forall a,b \in E$.

Time $\sim m \log n / \alpha^2$ for α accuracy.

Idea: $\mathrm{Reff}(a,b) = (\delta_a-\delta_b)^T L^+ (\delta_a-\delta_b) = \| L^{+1/2}(\delta_a-\delta_b)\|$

Johnson-Lindenstrauss: For any set of m vectors $x_1 \ldots x_m \in \mathbb{R}^n$

If R is $d \times n$ matrix of iid entries $\|Rx_i\|^2 \underset{1\pm\alpha}{\approx} \|x_i\|^2 \cdot \text{const}$ $\qquad d = \frac{\ln m}{\alpha^2}$

So, just need $RL^{+1/2}\delta_a$ for each a. Can compute quickly ...

Nikhil - how got thesis

Batson - Yale undergrad back then.