CPSC 462/562 Dan Spielman

Spectral and Algebraic Graph Theory

eigenvalues
eigenvectors

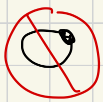
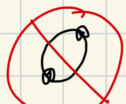
linear equations
operators

Goals: intro
get used to notation
interrupt/
ask questions

Graphs: $G = (V, E)$ E is set of pairs of elements of V

Write edges as (a,b)

Undirected: $(a,b) = (b,a)$

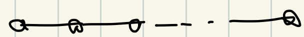
No self loops  or multi-edges 

Should have weights. Default weight $= 1$

Sources: social networks
communication
road

modeling: PPI

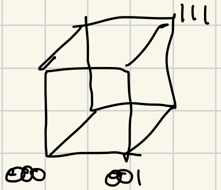
Abstract

path: 

$V = \{1, \dots, n\}$

$E = \{ (a, a+1) : 1 \le a < n \}$

ring: path + edge $(1, n)$

Hypercube 

$V = \{0,1\}^d$

$(a,b) \in E$ if $\left| \{ i : a(i) \neq b(i) \} \right| = 1$

Random: Edge (a,b) included with prob p, iid.

Adjacency matrix M. row/cols labeled by V

$$M(a,b) = \begin{cases} 1 & \text{if } (a,b) \in E \\ 0 & \text{o.w.} \end{cases}$$

Is a spreadsheet. Surprising if useful.

Diffusion Operator / Walk Matrix

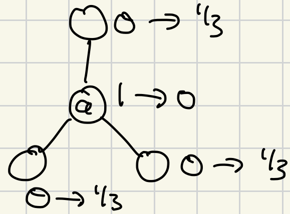
let D = diagonal matrix of degrees $\quad d = M\cdot\mathbb{1}, \; D = \mathrm{diag}(d)$

$$W = MD^{-1}$$

multiply by W. Let δ_a = elem unit in direction a

$$D^{-1}\delta_a = \frac{1}{d(a)}\delta_a$$

$$(MD^{-1}\delta_a)(b) = \begin{cases} \frac{1}{d(a)} & \text{if } (a,b) \in E \\ 0 & \text{o.w.} \end{cases}$$



If $p(a)$ = amount of stuff at a

Wp = amount after distributing among neighbors

Stuff is conserved. Total $= \mathbb{1}^T p = \sum_a p(a)$

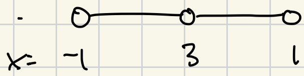
$$\mathbb{1}^T M D^{-1} p = d^T D^{-1} p = \mathbb{1}^T p$$

If p is prob. dist., Wp = after one step of random walk

Laplacian $L = D - M$. For $x \in \mathbb{R}^V$, aka $x: V \to \mathbb{R}$

$$x^T L x = \sum_{(a,b)\in E} (x(a)-x(b))^2 \, w_{a,b}$$

eg.



$x^T L x = 16 + 4 = 20$

ψ is an eigvec of M of eigval μ if $M\psi = \mu \psi$, $\psi \neq 0$

Thm Every real symmetric n-by-n matrix M

has n real eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$

and n orthonormal eigenvectors $\psi_1, \ldots, \psi_n$

s.t. $M\psi_i = \mu_i \psi_i$ $\qquad \psi_i^T \psi_j = \begin{cases} 1 & i=j \\ 0 & \text{o.w.} \end{cases}$

$Q = \begin{pmatrix} | & & | \\ \psi_1 & \cdots & \psi_n \\ | & & | \end{pmatrix}$ is orthogonal, $QQ^T = Q^T Q = I$

$\psi_1, \ldots, \psi_n$ not uniquely defined. $\qquad Q^T Q = I \Rightarrow Q^T = Q^{-1}$

Jupyter

Logistics

Pre-reqs: linear algebra, graphs, proof-based
endurance, some probability

Readings: my book, my notes

Work: 6 homeworks. Groups of up to 3.

No tests / exams

Proving -- no coding

562 = 462 + extra problems

Recommended exercises: At end of Chapter 1

Topics:

Graph structure: cuts, coloring, independent sets

The Zoo: fundamental examples

Estimating eigenvalues

Random Walks

Physical models: springs / resistors

Expanders, and applications: PSRG, ECC

Sparsification

Solving lin equs, computing eigenvectors

The Rayleigh quotient of x w.r.t. M is $\dfrac{x^T M x}{x^T x}$

Thm If M is symetric and x maximizes $\dfrac{x^T M x}{x^T x}$

Then $Mx = \mu_1 x$. Note $\psi_1^T M \psi_1 = \mu_1 \psi_1^T \psi_1$.

Expand $x = \sum_i c_i \psi_i$, where $c_i = \psi_i^T x$

proof $\sum_i c_i \psi_i = \sum_i (\psi_i^T x) \psi_i = \sum_i \psi_i (\psi_i^T x) = \sum_i (\psi_i \psi_i^T) x = Ix = x$

recall: $x(i) = \delta_i^T x$, $x = \sum_i x(i) \delta_i$

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claim: $x^T M x = \sum_i c_i^2 \mu_i$

proof $x^T M x = \left(\sum_i c_i \psi_i\right)^T M \left(\sum_j c_j \psi_j\right)$

$= \left(\sum_i c_i \psi_i\right)^T \left(\sum_j c_j \mu_j \psi_j\right)$ $\qquad \psi_i^T \psi_j = \begin{cases} 1 & i=j \\ 0 & \text{o.w.} \end{cases}$

$= \sum_i c_i^2 \mu_i$

proof of theorem ϑ_x

$$\frac{x^T M x}{x^T x} = \frac{\sum_i c_i^2 \mu_i}{\sum_i c_i^2} \leq \frac{\sum_i c_i^2 \mu_1}{\sum_i c_i^2} = \mu_1$$

equality iff $c_i = 0$ when $\mu_i < \mu_1$