Bipartite Expanders.

A d-regular bipartite graph has adjacency eigvals d and $-d$.

It is an ε-expander if all others have absolute value $\leq \varepsilon d$.

let $K_{n,n}$ be complete bipartite with $2n$ vertices

G is a bipartite ε-expander iff

$$(1-\varepsilon)\frac{d}{n} L_{K_{n,n}} \preceq G \preceq (1+\varepsilon)\frac{d}{n} L_{K_{n,n}}$$

As $K_{n,n}$ has eigvals n : eigvec $\begin{pmatrix}1\\1\end{pmatrix}$

$-n$: eigvec $\begin{pmatrix}1\\-1\end{pmatrix}$

and all the rest are 0.

How do we get a bipartite ε-expander?

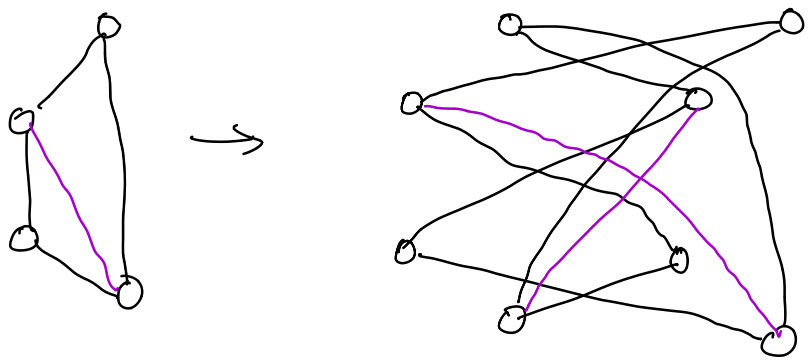
1. By direct construction: is easier than non-bipartite kind.

2. By the double-cover of an ε-expander.

Def The double-cover of a graph $G = (V, E)$ has vertices $V \times \{0,1\}$ and edges

$$((a,0),(b,1)) \text{ for all } (a,b) \in E.$$

Example



Prop If H is the double-cover of G then the adjacency eigenvalues of H are $\pm \mu_i$, where μ_i are adjacency eigenvalues of G.

Proof. $M_H = \begin{pmatrix} 0 & M_G \\ M_G & 0 \end{pmatrix}$

So, if G is an ε-expander H is a bipartite ε-expander.

Code construction

let vertex sets of G be $\mathcal{U}$ and V, $|\mathcal{U}|=n$

G has dn edges.

We put one bit on each edge, so

code length = dn.

It remains to impose constraints on those bits.

Assume that we have a linear code

C_0 of codeword length d

rate r_0 and

min relative distance δ_0.

We require that for all $a \in \mathcal{U} \cup V$

$$\left(Y(a,b)\right)_{(a,b)\in E} \in C_0.$$

So, the bits on edges attached to each vertex look like a codeword in that small code.

There is a matrix M_0 s.t. $x \in C_0$ iff $M_0 x = 0$

M_0 is $d(1-r_0) \times d$ ... it imposes $d - r_0 d$ linear constraints.

In total, the vertices impose $2nd(1-r_0)$ linear constraints.

This leaves at least $dn - 2nd(1-r_0) = nd(2r_0 - 1)$ degrees of freedom,

So the rate is $\geq 2r_0 - 1$. We need $r_0 \geq \frac{1}{2}$

Encoding: One can use linear algebra to construct a big generator matrix.
Encode by multiplying the message vector by this matrix.

Distance bound comes from expansion.

Theorem Let $G = (U \cup V, E)$ be a bipartite ε-expander, then for all $S \subseteq V$ and $T \subseteq U$,

$$\left| E(S,T) - \frac{d}{n}|S||T| \right| \le \varepsilon d \sqrt{|S|\,|T|}$$

Cor For $S \subseteq V$, $|S| = \sigma n$, $T \subseteq U$, $|T| = \tau n$
average degree of $G(S \cup T)$ – induced subgraph – is at most

$$\frac{2d\sigma\tau}{\sigma+\tau} + \varepsilon d$$

Proof ave degree $= \dfrac{2 \cdot \#\text{ edges}}{\#\text{vertices}}$

$$\le \underbrace{2\frac{\frac{d}{n}|S|\,|T|}{|S|+|T|}}_{= \frac{2d\sigma\tau}{\sigma+\tau}} + \underbrace{\frac{2\varepsilon d\sqrt{|S|\,|T|}}{|S|+|T|}}_{\le \varepsilon d \text{ as } 2\sqrt{\sigma\tau} \le \sigma+\tau}$$

Minimum Distance

Thm If $\varepsilon \le \delta_0/2$, then min rel dist δ of C satisfies $\delta \ge \delta_0^2/2$

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proof. Suffices to prove minimum weight of nonzero codeword $\ge dn \cdot \delta_0^2/2$.

Identify a codeword with the set of edges $F \subseteq E$ on which it is 1. $|F| = \phi dn$. Let F corresp a codeword.

Let $S \subseteq U$ and $T \subseteq V$ be endpoints of edges in F.

As min dist of C_0 is $\ge \delta_0 d$,

every vertex in U, V attached to an edge of F is attached to at least $\delta_0 d$.

$|S|, |T| \le \dfrac{|F|}{\delta_0 d}$. $|S| = \sigma n$, $|T| = \tau n$ gives

$\sigma, \tau \le \dfrac{\phi}{\delta_0}$

Ave degree of $G(S \cup T) \le \dfrac{2 d \sigma \tau}{\sigma + \tau} + \varepsilon d$

as $2\sigma\tau \le \sigma^2 + \tau^2 \le \frac{\phi}{\delta_0}(\sigma + \tau)$

ave degree of $G(S \cup T) \le d \frac{\phi}{\delta_0} + \varepsilon d$

As ave degree $\ge \delta_0 d$,

$\delta_0 \le \dfrac{\phi}{\delta_0} + \varepsilon \Rightarrow \delta_0^2 \le \phi + \varepsilon \delta_0$

If $\varepsilon \le \delta_0/2 \Rightarrow \phi \ge \frac{\delta_0^2}{2}$.

Decoding

U-step: for each $a \in U$, map bits on attached edges to closest codeword.

V-step: same

<u>Thm</u> If $\varepsilon \le \delta_0/3$, this alg corrects $dn \frac{\delta^2}{18}$ errors after $\lg_{\frac{4}{3}} n$ iterations.

<u>Lem1</u> $F \subset E$,

$S \subset U$ endpoints of F,

$T \subset V$ endpoints of $\ge \frac{\delta_0 d}{2}$ edges of F.

If $|S| \le \delta n/9$ then $|T| \le \frac{3}{4}|S|$

<u>proof</u> ave degree of $G(S \cup T) \ge \frac{\delta_0 d}{2} \cdot 2 \cdot \frac{|T|}{|S|+|T|}$

$$= \frac{\delta_0 d \tau}{\sigma + \tau}$$

By Cor, $\le \frac{2d\sigma\tau}{\sigma+\tau} + \varepsilon d$

$$\Rightarrow \tau \le \frac{\varepsilon\sigma}{\delta_0 - 2\sigma - \varepsilon}$$ use $\varepsilon \le \frac{\delta_0}{3}$, $\sigma \le \delta_0/9$

$$\tau \le \frac{3}{4}\sigma$$

<u>lem 2</u> let F be edges in error after a U-step.

S = $a \in U$ attached to F.

let T = $b \in V$ attached to edges in error after a V-step.

If $|S| \le \frac{\alpha n \delta_0}{9}$, $|T| \le \frac{3}{4}|S|$.

<u>proof.</u> each $b \in T$ is attached to at least $\frac{\delta_0 d}{2}$

edges of F.

apply lem 1.

<u>proof of Thm</u> let F be edges initially in error.

$|F| \le \frac{\alpha n \delta_0^2}{18}$

S = vertices attached to error edges after first

U-step. $|S| \le \frac{|F|}{\frac{\delta_0 d}{2}} = \frac{\delta_0 n}{9}$

Now, lem 2 size of set in error decreases

by $\frac{3}{4}$ each iteration.