2025-Mar-31. Constructing Expanders.

Will say a d-regular graph G is an ε-expander if $|\lambda_i - d| \le \varepsilon d$ $\forall i \ge 2$ $\Leftrightarrow$ $|\mu_i| \le \varepsilon d$

Our goal today: $n \to \infty$, d constant, $\varepsilon < 1$ constant. Generalized hypercubes have $d = \log n$, so no good.

Today, measure spectral gap $\delta = 1 - \varepsilon$, so want $\delta > 0$. $\delta d \le \lambda_i \le (2-\delta) d$

Is known that random regular graphs are expanders with $\delta \sim 1 - \frac{2\sqrt{d-1}}{d}$, but not explicit.

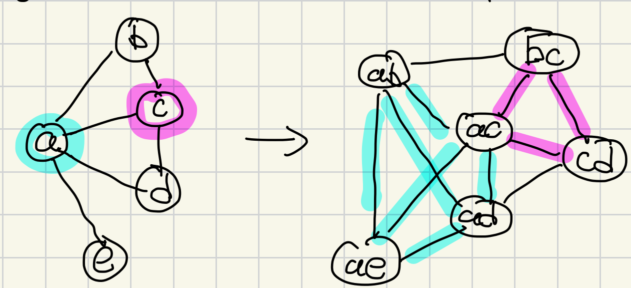
And, want very explicit. Given name of node, can compute neighbors in time poly(#bits in name)
Can then use in PSRG.

Recursive construction.

	n	d	δ
line-graph	$n \to nd/2$	$d \to 2(d-1)$	$\delta \to \delta/2$
clique → expander	—	$\downarrow k \ll d$	$\delta \to \delta(1-\alpha)$
squaring	—	$d \to d(d-1)$	$\delta \to 2\delta - \delta^2$

Let $G = (V, E)$. The line-graph of G, $H = (E, F)$
with $((a,b),(b,c)) \in F$ if $a \ne c$.

Edges of E connected if share endpoint



squaring: G^2 = edge (a,b) s.t. $\mathrm{dist}_G(a,b) = 2$
i.e. if $\exists\, c$ s.t. (a,c) and $(c,b) \in E$

$$M_{G^2} = M_G - dI$$

Thm If H is line graph of a d-regular G then H is $2(d-1)$ regular, and has all eigenvalues of G and equal $2d$ with mult. $\frac{dn}{2}-n$.

Recall $L_G = U^T U$, where U is signed edge-vertex adj matrix. $U((a,b),c) = \begin{cases} 1 & \text{if } c=a \\ -1 & \text{if } c=b \\ 0 & \text{o.w.} \end{cases}$

Let $B = |U|$, so only 0/1. $B^T B = |L_G| = D + M_G$

$BB^T = M_H + 2I$

proof. $B^T B$ and BB^T same eigvals up to zeros

λ_i an eigval of $L_G = dI - M_G \Leftrightarrow$

$d - \lambda_i \in \text{eigs}(M_G) \Leftrightarrow 2d - \lambda_i \in \text{eigs}(dI + M_G)$

$\Leftrightarrow 2d - \lambda_i \in \text{eigs}(M_H + 2I)$ 0

$\Leftrightarrow 2(d-1) - \lambda_i \in \text{eigs}(M_H)$ -2

$\Leftrightarrow \lambda_i$ an eigval of $L_H = 2(d-1)I - M_H$ $2d$

Eigvals stay same, but degree almost doubles

so gap $\delta \to \delta/2$

Every $a \in G \to d$-clique in H

$$H = \sum_{a \in V} K_a \quad \text{where } K_a \text{ is clique on the edges } (a,b)$$

If G has edge of weight w, create w edges in H, joined by edges of weight 2

Replace d-cliques by expanders.

let Z be k-regular α-expander on d nodes

$$\hat{H} = \sum_{a \in V} Z_a \quad Z_a = \text{copy of } Z \text{ on same vertices as } K_a$$

$$(1-\alpha) K_a \preccurlyeq \tfrac{k}{d} Z_a \preccurlyeq (1+\alpha) K_a$$

$\to \hat{H}$ has degree $2k$, $\frac{nd}{2}$ vertices

and gap $\geq (1-\alpha)\,\delta/2$

To improve gap, square graph. For d-regular G, G^2 = graph s.t. $M_{G^2} = M_G^2 - dI$

(a,b) is edge of G^2 if $\exists c$ s.t. (a,c) and (c,b) edges of G. weight = # of such c.
remove d self-loops

$\mu_i \in \text{eigs}(M_G) \Rightarrow \mu_i^2 - d \in \text{eigs}(M_{G^2})$. G^2 has degree $d(d-1)$

$\boxed{\text{Lem } \delta(G^2) \geq 2\delta(G) - \delta(G)^2}$ eg. if $\delta \leq 1/3$, $2\delta - \delta^2 \geq 5/6\,\delta$. Proof $\mu_i = d - \lambda_i$, $\lambda_i \geq \delta d$

$\text{eigs}(L_{G^2}) = d(d-1) - (\mu_i^2 - d) = d^2 - (d-\lambda_i)^2 = 2d\lambda_i - \lambda_i^2 \geq 2d^2\delta - d^2\delta^2 = d^2(2\delta - \delta^2) \geq d(d-1)(2\delta - \delta^2)$

$(\hat{H})^2$ has $\frac{dn}{2}$ vertices, $2k(2k-1)$-regular, gap $\geq 2(1-\alpha)\delta/2 - (1-\alpha)^2(\delta/2)^2 \approx (1-\alpha)\delta$

To get above δ, use $((\hat{H})^2)^2$. If $\delta(G) \geq 1/3$, $\alpha \leq 1/3$, $\delta(((\hat{H})^2)^2) \geq 1/3$

Has $\frac{dn}{2}$ vertices, degree approx $(4k)^4$, gap $> \delta$

Start with any G_0 that satisfies these conditions, like K_n, $n \geq (4k)^4$, $\delta_0 \geq 1/3$

Need $\mathcal{Z}$, expander on $(4k)^4$ vertices, degree k. α-expander, $\alpha \leq 1/3$ would suffice.

so, use k s.t. $\frac{2\sqrt{k-1}}{k} \sim 1/3$, $k \sim 36$