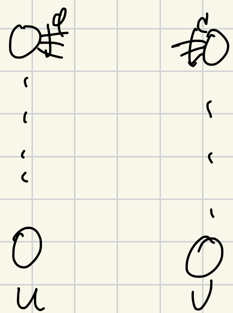
2025-Mar-26. Expander Codes

Idea: Bipartite Expander Graph $G=(U \cup V, E)$ d-regular

$|U|=|V|=n$

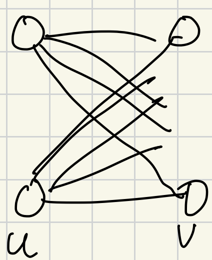


Bits on edges, so code of length $|E|=dn$

Constraints come from a code C_0 of length d. (linear)

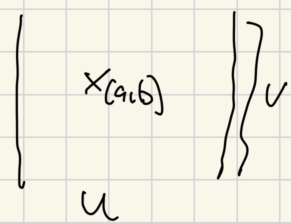
For each vertex, require bits on its edges to be a codeword in C_0

Ex. $|U|=|V|=d$, complete bipartite. If $d=7$, could use Hamming code for C_0



For $a \in U$, $b \in V$ is edge (a,b) and bit $x_{(a,b)}$.

Arrange in a grid.



← each row and column must be codeword in C_0

Are there any codewords? Yes C_0 is linear, so all-0 is codeword

Let r_0 = rate of C_0. So, C_0 has dimension dr_0, determined by $d(1-r_0)$ constraints.

Are $2n$ vertices. Impose $2nd(1-r_0)$ linear constraints.

Are dn variables, so space of solution has dimension $\geq dn - 2dn(1-r_0)$

$= dn(2r_0 - 1)$. Rate $r = 2r_0 - 1$. Good if $r_0 > \frac{1}{2}$, like Hamming code $\frac{4}{7}$

Note do not expect params of big codes to be as good as they are for small codes.

Get min distance and decoding from expansion.

A d-regular bipartite ε-expander approximates $\frac{d}{n} K_{n,n}$ - $K_{n,n}$ = complete bipartite

$(1-\varepsilon)\frac{d}{n} K_{n,n} \preccurlyeq G \preccurlyeq (1+\varepsilon)\frac{d}{n} K_{n,n}$. Eigvals of G $\lambda_0 = 0$, $\lambda_{2n} = 2d$, $|\lambda_i - d| \leq \varepsilon d$

<u>Theorem</u> For all $S \subseteq U$, $U \subseteq T$, let $E(S,T)$ = #edges between S and T

$\left| E(S,T) - \frac{d}{n}|S||T| \right| \leq \varepsilon d \sqrt{|S||T|}$

<u>Cor</u> Average degree of $G(S \cup T) \leq \frac{2d\sigma\tau}{\sigma+\tau} + \varepsilon d$ where $|S| = \sigma n$, $|T| = \tau n$

proof of Cor ave degree $= \frac{2 \# \text{edges}}{\# \text{vertices}} \le \frac{2\frac{d}{n}|S||T| + 2\varepsilon d\sqrt{|S||T|}}{|S|+|T|} = \frac{2d\sigma\tau + 2\varepsilon d\sqrt{\sigma\tau}}{\sigma+\tau}$

So, a set of few edges must attach to vertex with few of them

$2\sqrt{\sigma\tau} \le \sigma + \tau \rightarrow \le \frac{2d\sigma\tau}{\sigma+\tau} + \varepsilon d$

Thm If $\varepsilon \le \delta_0/2$, then min rel dist of code $\ge \delta_0^2/2$

proof Linear $\rightarrow$ consider min wt of nonzero codeword

codeword $\leftrightarrow$ set of edges of which 1. $F \subseteq E$. let $|F| = \phi d n$ wts $\phi \ge \delta_0^2/2$

S, T = endpoints of edges in F. Min dist of $C_0 \ge \delta_0 d \rightarrow$ each vertex in S, T touches $\ge \delta_0 d$

$\rightarrow |S|, |T| \le \frac{|F|}{\delta_0 d}$ $\quad \sigma, \tau \le \frac{\phi}{\delta_0}$

$d\delta_0 \le$ Ave degree of $G(S \cup T) \le \frac{2d\sigma\tau}{\sigma+\tau} + \varepsilon d$. $\quad 2\sigma\tau \le \sigma^2 + \tau^2 \le \frac{\phi}{\delta_0}(\sigma+\tau)$

$\delta_0 \le \frac{\phi}{\delta_0} + \varepsilon \qquad \varepsilon \le \delta_0/2 \rightarrow \delta_0/2 \le \phi/\delta_0 \leftrightarrow \phi \ge \delta_0^2/2$

To decode : U-step : apply decoding on C_0 for all vertices in U

V-step : decode for V

until all vertices see codewords, or give up

Thm If $\varepsilon \le \delta_0/3$, this alg corrects $dn\,\delta^2/18$ errors in $\log_{4/3} n$ iterations

lem let $F \subset E$ be bits in error after a U-step. $S \subset U$ nbrs of F, $T \subset V$ = nodes touching $\ge \frac{\delta_0 d}{2}$ of F
If $|S| \le \delta_0 n/9$ then $|T| \le \frac{3}{4}|S|$.
Only nodes in T produce errors in U-step. So # nodes attached to errors decreases.

proof: $\frac{2d\sigma\tau}{\sigma+\tau} + \varepsilon d \ge$ Ave degree $G(S \cup T) \ge |T| \frac{\delta_0 d}{2} \cdot \frac{2}{|S|+|T|} = \frac{\delta_0 d \tau}{\sigma+\tau}$

$\rightarrow \quad 2\sigma\tau + \varepsilon\sigma + \varepsilon\tau \ge \delta_0 \tau \iff \tau \le \frac{\varepsilon\sigma}{\delta_0 - 2\sigma - \varepsilon}$

$\varepsilon \le \delta_0/3, \quad \sigma \le \delta_0/9$

$\tau \le \frac{3}{4}\varepsilon$

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proof of Thm To start, If F edges initially in error, $|F| \le dn\,\delta_0^2/18$

S = vertices attached to errors after first U-step, $|S| \le \frac{|F|}{\delta_0 d/2} \le n\delta_0^2/9$

So, can apply lem.