2025-March-3

Expanders: d-regular, d constant. Small $S \subseteq V$ have many neighbors $|N(S)| \geq \frac{d}{5}|S|$

Approximate $\frac{d}{n}K_n$. Pseudo-random properties. Random d-regular are expanders.

__Def.__ A d-regular graph G is an ε-expander if $\|L_G - \frac{d}{n}L_{K_n}\| \leq \varepsilon d$

Is equivalent to $|\lambda_i - d| \leq \varepsilon d$ for all $i \geq 2$. Because $\lambda_i(\frac{d}{n}K_n) = d \quad \forall i \geq 2$.

In adjacency, is $\mu_1 = d$, $|\mu_i| \leq \varepsilon d \quad \forall i \geq 2$. Ramanujan: $\varepsilon \leq \frac{2\sqrt{d-1}}{d}$

For $A, B \subseteq V$, $A \cap B = \phi$ $\quad E(A,B) \hat{=} \{(a,b) \in E, a \in A, b \in B\}$

__Thm 1__ For all $A, B \subseteq V$, $A \cap B = \phi$, $|A| = \alpha n$, $|B| = \beta n$

$$\Big| |E(A,B)| - d\alpha\beta n \Big| \leq \varepsilon d n \sqrt{\alpha - \alpha^2}\sqrt{\beta - \beta^2}.$$ Close to $d\alpha\beta n$ when $\alpha\beta > \varepsilon$

Motivation: Consider choosing A and B at random, subject to $|A| = \alpha n$, $|B| = \beta n$

$\Pr[a \in A, b \in B] \approx \alpha\beta$, so $\Pr[(a,b) \in E(A,B)] \approx 2\alpha\beta$ and $\mathbb{E}[|E(A,B)|] \approx \frac{dn}{2} \cdot 2\alpha\beta = d\alpha\beta n$

For K_n, $|E(A,B)| = (\alpha n)(\beta n)$, so $w(E_{\frac{d}{n}K_n}(A,B)) = d\alpha\beta n$.

proof of Thm 1 $\mathbb{1}_A^T L_G \mathbb{1}_B = \mathbb{1}_A^T (dI - M_G)\mathbb{1}_B = 0 - |E(A,B)| = -|E(A,B)|$

Let $H = \frac{d}{k} K_n$. $\mathbb{1}_A^T L_H \mathbb{1}_B^T = -d\alpha\beta n$.

And, $|\mathbb{1}_A^T (L_G - L_H)\mathbb{1}_B| \le \|\mathbb{1}_A^T\| \cdot \|(L_G - L_H)\mathbb{1}_B\| \le \sqrt{\alpha n} \cdot \varepsilon d \cdot \sqrt{\beta n} = \varepsilon d \sqrt{\alpha\beta}\, n$

$\rightarrow \big|\, |E(A,B)| - d\alpha\beta n \big| \le \varepsilon d n \sqrt{\alpha\beta}$

To improve, replace $\mathbb{1}_A$ with $X_A = \mathbb{1}_A - \alpha\mathbb{1}$, $X_B = \mathbb{1}_B - \beta\mathbb{1}$, $\|X_A\| = \sqrt{\alpha - \alpha^2}\, n$

$X_A^T L_G X_B = \mathbb{1}_A^T L_G \mathbb{1}_B$, so $\big| |E(A,B)| - d\alpha\beta n \big| \le \varepsilon d n \sqrt{(\alpha - \alpha^2)(\beta - \beta^2)}$

For $A \subset V$, define $N(A) = \{b : \exists a \in A, (a,b) \in E\}$ $\left(\approx \frac{\alpha}{\varepsilon^2} \text{ for } \alpha \text{ small}\right)$

<u>Tanner's Thm</u> For $|A| = \alpha n$, $|N(A)| = \gamma n$, $\gamma \ge \frac{\alpha}{\varepsilon^2(1-\alpha) + \alpha}$ $\left(\text{for } \varepsilon \approx \frac{2\sqrt{d-1}}{d}, \approx \alpha \frac{d}{4}\right)$

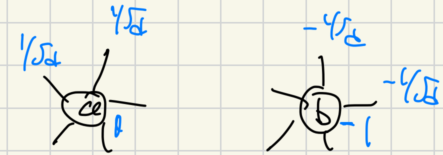
<u>proof</u> Let $B = V - N(A)$, so no edges between A and B. $|B| = \beta n$ $|N(A)| = V - B$, $\gamma = 1 - \beta$

By Thm 1: $\alpha\beta d n \le \varepsilon d n \sqrt{\alpha\beta(1-\alpha)(1-\beta)} \iff \sqrt{\alpha\beta} \le \varepsilon\sqrt{(1-\alpha)(1-\beta)} \iff \alpha\beta \le \varepsilon^2(1-\alpha)(1-\beta)$

$\iff \alpha(1-\gamma) \le \varepsilon^2(1-\alpha)\gamma \iff \alpha \le \varepsilon^2(1-\alpha)\gamma + \alpha\gamma \iff \gamma \ge \frac{\alpha}{\varepsilon^2(1-\alpha) + \alpha}$

Ramanujan Bound (Alon–Boppana). For large n, $\lambda_2 \le d - 2\sqrt{d-1}$ + small

Want $x \perp 1$ so that $x^T L x \le d - 2\sqrt{d-1}$ + small

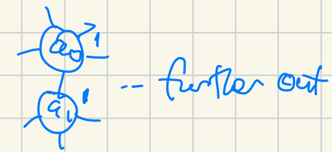


Idea: let $a, b \in V$, no edges between $a \cup N(a)$ and $b \cup N(b)$

$$x(c) = \begin{cases} 1 & c = a \\ 1/\sqrt{d} & c \in N(a) \\ -1 & c = b \\ -1/\sqrt{d} & c \in N(b) \end{cases}$$

$$x^T L x = 2\left[d\left(1 - 1/\sqrt{d}\right)^2 + d(d-1)\left(\tfrac{1}{\sqrt{d}}\right)^2\right] = 2\left[d - 2\sqrt{d} + 1 + d - 1\right] = 4(d - \sqrt{d})$$

$$x^T x = 2\left(1 + d \cdot \tfrac{1}{d}\right) = 4 \qquad \frac{x^T L x}{x^T x} \le d - \sqrt{d}$$



Thm If are edges (a_0, a_1) and (b_0, b_1) at dist $\ge 2k+2$, $\lambda_2 \le d - 2\sqrt{d-1} + \frac{2\sqrt{d-1} - 1}{k+1}$

proof def $A_0 = \{a_0, a_1\}$ $A_1 = N(A_0) - A_0$, ..., A_k where $A_i = N(A_{i-1}) - \bigcup_{j<i} A_j$. Same for B.

for $a \in A_i$, $x(a) = (d-1)^{-i/2}$ $a \in B_i$, $x(a) = -\beta (d-1)^{-i/2}$. Choose β so $x \perp \mathbb{1} = 0$.

$$A\text{-denom} = \sum_{i=0}^{k} |A_i| (d-1)^{-i} \qquad A\text{-numer} \le \sum_{i=0}^{k-1} |A_i| (d-1)\left((d-1)^{-i/2}\left(1 - 1/\sqrt{d-1}\right)\right)^2 + |A_k|(d-1)(d-1)^{-k}$$

$(d-1)\left(1 - 1/\sqrt{d-1}\right)^2 = d - 2\sqrt{d-1}$

$$= \sum_{i=0}^{k-1} |A_i| \frac{d - 2\sqrt{d-1}}{(d-1)^i} + \frac{|A_k|(d-1)}{(d-1)^k} = \sum_{i=0}^{k} |A_i| \frac{d - 2\sqrt{d-1}}{(d-1)^i} + |A_k| \frac{2\sqrt{d-1} - 1}{(d-1)^k}$$

$$\sum_{i=0}^{k} |A_i| \frac{d-2\sqrt{d-1}}{(d-1)^i} \le (d-2\sqrt{d-1})\, A\text{-denom} \quad \text{and} \quad |A_k| \le (d-1)^{k-i}|A_i| \to |A_k| \le \frac{1}{k+1}\sum_{i=0}^{k}|A_i|(d-1)^{k-i}$$

$$\to |A_k| \frac{2\sqrt{d-1}-1}{(d-1)^k} \le \frac{2\sqrt{d-1}-1}{k+1}\sum_{i=0}^{k}|A_i|(d-1)^{-i} \le \frac{2\sqrt{d-1}-1}{k+1}(A\text{-denom})$$

So, $\dfrac{x^T L x}{x^T x} = \dfrac{A\text{-numer} + B\text{-numer}}{A\text{-denom} + B\text{-denom}} \le \max\left(\dfrac{A\text{-numer}}{A\text{-denom}}, \dfrac{B\text{-numer}}{B\text{-denom}}\right) \le d-2\sqrt{d-1} + \dfrac{2\sqrt{d-1}-1}{k+1}.$