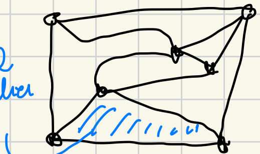
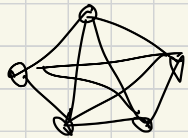
2025-Feb-26

$G=(V,E)$ is planar if can draw in plane with no crossing edges

face: connected region when remove edges



planar



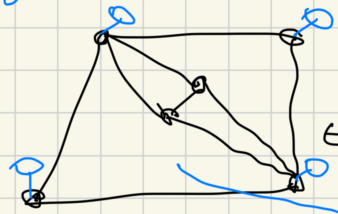
not planar

$\exists$ map $x: V \to \mathbb{R}^2$, $a \neq b \to x(a) \neq x(b)$

$x: (a,b) \to$ curve from a to b in $\mathbb{R}^2$

interiors of curves do not cross or contain vertices.

Can draw with straight lines, and springs.



← issue for springs — not 3 connected.

vertices bounding faces not uniquely determined

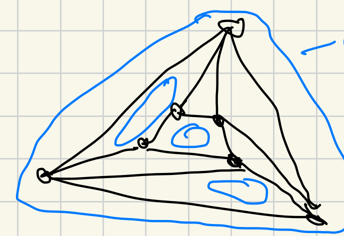
G is 3-connected if $\forall S \subseteq V$, $|S| \leq 2$, $G(V-S)$ is connected.

A 3-connected planar graph has a unique set of faces $F_1, \dots, F_m$, that are cycles in G
each edge is in exactly two faces

In every planar drawing
the faces are the boundaries of the regions obtained when remove the edges and vertices

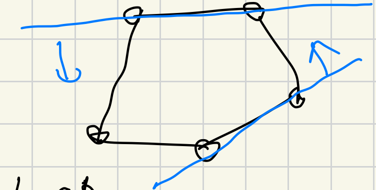
Tutte: if fix vertices on a face to corner of a convex polygon,
let rest be harmonic, get a planar drawing of G

Harmonic $\Rightarrow$ all other vertices are strictly inside.



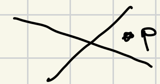
Def $a_1, \dots, a_k$ are corners of a convex polygon, in order,

if all vertices are on one side of line through a_i, a_{i+1}



Lem 1: For every non-boundary (a, b), its faces are on opposite sides of line through ab

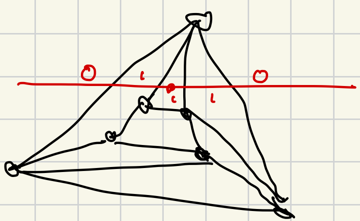
Lem 1 → Thm Assume bwoc two edges cross

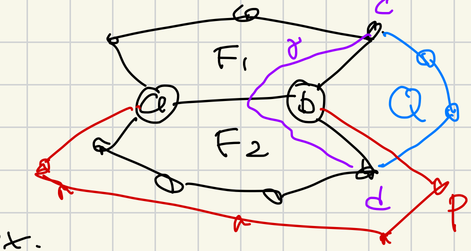


then is a point p inside ≥ 2 faces

To show can not happen, let l be a line through p that does not intersect vertices or edge crossings.

Show that on line, # faces containing each point ≤ 1. At boundary $= 1$. when cross edge stays 1





Lem 2 For 3-con planar graph G - let (a,b) be in faces F_1 and F_2.
Let Q be a path connecting F_1 to F_2, $a,b \notin Q$
only touches F_1, F_2 at endpoints.

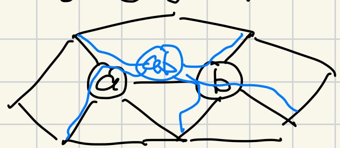
If P is a path from a to b, $P \neq (a,b)$ then P intersects Q at a vertex.

Proof: let $c = Q \cap F_1$, $d = Q \cap F_2$. Draw a curve γ from c to d inside F_1 and F_2, crosses ab

So $\gamma \cup Q$ is a closed curve that separates a from b in plane.
$\to$ P must cross Q, and must do so at a vertex.

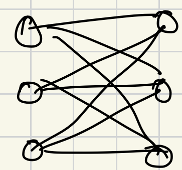
Facts If contract an edge (a,b) replace a and b with vertex ab, with nbrs of a and b,

still planar.

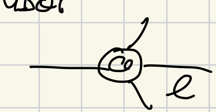


K_5 and $K_{3,3}$ are not planar

$K_{3,3}$



Prop 3 In a Tutte embedding, if a line l goes through a and a has a neighbor above l, then it has a neighbor below l. Consequence of harmonic.



Lem 4 In Tutte embedding, let H be a half-space = line and all points on one side of it.

Let S = vertices in H. Then all vertices in S are connected by paths lying entirely in S.

i.e. $G(S)$ is connected.

proof. Let $H = \{x : \alpha^T x \geq \beta\}$ $f(x) = \alpha^T x - \beta$. Let $a \in H$. Will give non-decreasing inf path to B

If $\exists\, b \sim a$ so that $f(b) > f(a)$, can follow increasing path to boundary : harmonic at b

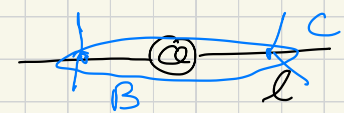
$\Rightarrow \exists\, c \sim b$ s.t. $f(c) > f(b)$, keep increasing until hit B

Otherwise, for all nbrs $b \sim a$, $f(b) = f(a)$. But is path from a to pos boundary, so eventually can increase

---

Lem 5 In a Tutte embedding, no vertex is colinear with all of its neighbors.

proof. Let ℓ be a line containing a and all of its neighbors. Let $B = \{b :$ can reach from a never leaving $\ell\} \cup \{a\}$

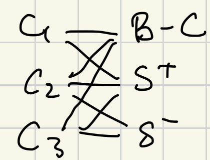


Let $C = \{b \in B$ s.t. b has neighbor not in $\ell\}$. 3-con $\Rightarrow |C| \geq 3$

To get contradiction, let S^+ = vertices above ℓ are connected, so contract all.

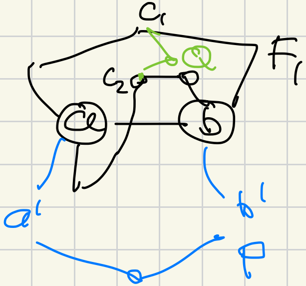
S^- be vertices below ℓ, contract.

Let $c_1, c_2, c_3 \in C$, contract $B - C$



$K_{3,3}$

Lem 1: Let F_1 and F_2 be the faces containing a non-boundary edge (a,b). Let l = line through $x(a)$ and $x(b)$. Then vertices of F_1 and F_2 lie on opposite sides of l and only a, b are on l.



proof Assume bwoc, $\exists\ c_1 \in F_1$ and $c_2 \in F_2$ both above or on l

Lem 4 $\Rightarrow$ $\exists$ path Q above line connecting c_1 to c_2

Lem 5 $\Rightarrow$ $\exists\ a' \sim a$, below line and $b' \sim b$ below line

Lem 4 $\Rightarrow$ a' and b' connected by path below line. $P = a \to a' \to \text{path} \to b' \to b$

Lem 2 $\Rightarrow$ P intersects Q, contradiction