2024-Feb-17 Random Walks

M = adjacency matrix $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$

$W = MD^{-1}$ walk matrix

$\tilde{W} = \frac{1}{2}I + \frac{1}{2}W$ Lazy walk $1 = \omega_1 \geq \omega_2 \geq \cdots \geq \omega_n$

$A = D^{-1/2}MD^{-1/2}$ Normalized Adjacency $= D^{-1/2}WD^{1/2}$

$N = D^{-1/2}LD^{-1/2} = I - A$ Normalized Laplacian

$0 = \nu_1 \leq \nu_2 \leq \cdots \leq \nu_n \quad \omega_i = 1 - \nu_i/2$

Random walk on weighted, connected G.

$\Pr[\text{move from } a \text{ to } b] = \dfrac{w_{a,b}}{d(a)}$

Track prob distribution. $p \geq 0$, $\mathbb{1}^T p = 1$

Usually $p_0 = \delta_s$. Start at $s \in V$

$p_{t+1}(b) = \sum_{a \sim b} p_t(a) \dfrac{w_{a,b}}{d(a)} \qquad p_{t+1} = MD^{-1}p_t$

$W = MD^{-1}$

W <u>not</u> symmetric. But $D^{-1/2}WD^{1/2} = D^{-1/2}MD^{-1/2} \triangleq A$ is

$A\psi = \lambda\psi \iff W(D^{1/2}\psi) = \lambda D^{1/2}\psi$

<u>proof</u>

$\lambda(D^{1/2}\psi) = D^{1/2}(\lambda\psi) = D^{1/2}(A\psi) = (MD^{-1})(D^{1/2}\psi)$

A and W have same eigenvalues

Warning: $\|W\| \neq$ largest eigenvalue.

Fact: $Ad^{1/2} = d^{1/2} \qquad \psi_1 = \dfrac{d^{1/2}}{\|d^{1/2}\|}$

$D^{-1/2}MD^{-1/2}d^{1/2} = D^{-1/2}M\mathbb{1} = D^{-1/2}d = d^{1/2}$

So, $d^{1/2}$ is Perron vector, $\lambda_{max} = 1$, $\lambda_{min} \geq -1$

Usually work with lazy walk

stays put with prob $1/2$ $\quad \tilde{W} = \frac{1}{2}I + \frac{1}{2}W$

Eigvals $1 = \omega_1 \geq \omega_2 \geq \cdots \geq \omega_n \geq 0$

Similarly, $Wd = MD^{-1}d = M\mathbb{1} = d$
$\tilde{W}d = \frac{1}{2}d + \frac{1}{2}d = d$

Stable distribution is $\pi \triangleq \frac{d}{\mathbb{1}^T d}$

$p_t = \tilde{W}^t p_0 \longrightarrow \pi$ as $t \to \infty$

proof Let $\psi_1, \ldots, \psi_n$ be orthonormal eigvecs of A

For any p_0, write $D^{-1/2}p_0 = \sum_i c_i \psi_i$, $c_i = \psi_i^T D^{-1/2} p_0$

$$c_1 = \frac{1}{\|d^{1/2}\|}\left(d^{1/2}\right)^T D^{-1/2} p_0 = \frac{1}{\|d^{1/2}\|}\mathbb{1}^T p_0 = 1/\|d^{1/2}\|$$

$$p_t = \tilde{W}^t p_0 = D^{1/2} D^{-1/2} \tilde{W}^t D^{1/2} D^{-1/2} p_0 = D^{1/2}\left(D^{-1/2}\tilde{W}D^{1/2}\right)^t D^{-1/2} p_0$$
$$= D^{1/2}\left(\tfrac{1}{2}I + \tfrac{1}{2}A\right)^t D^{-1/2} p_0 = D^{1/2} \sum_i \omega_i^t c_i \psi_i$$
$$= \underline{D^{1/2}\frac{1}{\|d^{1/2}\|}\psi_1} + \underbrace{\sum_{i\geq 2}\omega_i^t c_i \psi_i}_{\to 0 \text{ as } \omega_i \in [0,1)}$$
$$\hookrightarrow = D^{1/2} d^{1/2}/\|d^{1/2}\|^2 = d/\sum_a d(a) = \pi$$

Thm 1 $\forall a, b \in V$. If $p_0 = \delta_a$, then $\forall b$ $|\pi(b) - p_t(b)| \leq \sqrt{d(b)/d(a)}\ \omega_2^t$

proof $p_t(b) = \delta_b^T p_t = \pi(b) + \delta_b^T D^{1/2} \sum_{i\geq 2} \omega_i^t c_i \psi_i$, $c_i = \psi_i^T D^{-1/2}\delta_a = \frac{1}{\sqrt{d(a)}}\psi_i^T \delta_a$, $\delta_b^T D^{1/2} = \sqrt{d(b)}\,\delta_b$

$$\left|\delta_b^T \sum_{i\geq 2}\omega_i^t \psi_i \psi_i^T \delta_a\right| = \left|\sum_{i\geq 2}\omega_i^t (\delta_b^T\psi_i)(\psi_i^T\delta_a)\right| \leq \sum_{i\geq 2}\omega_i^t |\delta_b^T\psi_i||\psi_i^T\delta_a| \leq \omega_2^t \sum_{i\geq 2}|\delta_b^T\psi_i|\cdot|\psi_i^T\delta_a|$$
$$\leq \omega_2^t \sum_{i\geq 1}|\delta_b^T\psi_i||\psi_i^T\delta_a| \leq \omega_2^t\left(\sum_{i\geq 1}(\delta_b^T\psi_i)^2\right)^{1/2}\left(\sum_{i\geq 1}(\psi_i^T\delta_a)^2\right)^{1/2} \text{ by C-S } \leq \omega_2^t\|\delta_a\|\,\|\delta_b\| \leq \omega_2^t$$

Say walk mixes by time t if $|\pi(b) - p_t(b)| \leq \frac{1}{2}\pi(b)$ $\forall b \in V$. $\frac{1}{2}$ arbitrary

Normalized Laplacian $N = D^{-1/2} L D^{-1/2} = I - A$ eigvals $0 \le \nu_1 \le \cdots \le \nu_n$ $\omega_i = 1 - \nu_i/2$

$\tilde{\omega} = I - \frac{1}{2} D^{1/2} N D^{-1/2}$. Thm 1 says $|P_t(b) - \pi(b)| \le \sqrt{d(b)/d(a)} \left(1 - \nu_2/2\right)^t$

Note $1-x \le e^{-x}$, $\forall x$ and $1-x \approx e^{-x}$ small x. $\pi(b) = \frac{d(b)}{d(V)}$. So, mixed by $t = \frac{2 \ln\left(\frac{2d(V)}{\sqrt{d(a)d(b)}}\right)}{\nu_2}$.

need $|P_t(b) - \pi(b)| \le \frac{1}{2}\frac{d(b)}{d(V)}$ $\quad \sqrt{d(b)/d(a)} \left(1 - \frac{\nu_2}{2}\right)^t \le \sqrt{d(b)/d(a)}\, e^{-\frac{\nu_2}{2}t} \le \frac{1}{2}\frac{d(b)}{d(V)}$

$\Longleftrightarrow e^{-\frac{\nu_2 t}{2}} \le \frac{1}{2}\frac{\sqrt{d(b)d(a)}}{d(V)}$ $\quad \Longleftrightarrow t \ge \frac{2}{\nu_2} \underbrace{\ln\left(\frac{2d(V)}{\sqrt{d(a)d(b)}}\right)}$. Focus on ν_2

often unnecessary

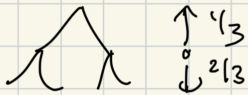
$\nu_2 = \min_{x \perp d} \frac{x^T L x}{x^T D x}$ $\quad$ So, $\frac{\lambda_2}{d_{max}} \le \nu_2 \le \frac{\lambda_2}{d_{min}}$

Ex. K_n $\lambda_2 = n$, $d_{max} = d_{min} = n-1$ $\nu_2 \approx 1$. Mix in $\log n$ steps of lazy walk ...
need $\log n$ to leave start vertex.

P_n: $\lambda_2 \sim \frac{c}{n^2}$, $\nu_2 \sim \frac{c}{n^2}$. If start in middle, stdev of |position| is $\sim \sqrt{t}$

So, need $t \ge cn^2$ to leave middle

CBT

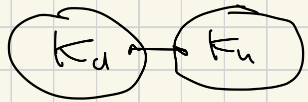


From root mix in ~d steps.

From leaf, need to hit root. Takes $\sim 2^d \sim n$ steps

$\lambda_2 = c/n, \quad \nu_2 = c/n$

Dumbbell.



If in this part, need $\sim n^2$ steps to go to right.
So, mixing time $\geq n^2$.

Test vec $+1$ -1 $\quad \lambda_2 \leq c/n, \quad \lambda_{max} = n \rightarrow \quad \nu_2 \leq c/n^2$

<u>Lem</u> If G is unweighted of diameter r, then $\lambda_2 \geq \frac{2}{r(n-1)}$

For Dumbbell, $r=3$, $\lambda_2 \geq \frac{2}{3(n-1)}$ $\quad \nu_2 \geq \frac{c}{n^2}$.

Proof. Let $P_{a,b}$ be path in G from a to b of length $\leq r$

$$K_n = \sum_{a<b} G_{a,b} \preccurlyeq r \sum_{a<b} P_{a,b} \preccurlyeq r \sum_{a<b} G = \binom{n}{2} r G$$

So, $\quad \lambda_2(K_n) = n \quad \leq \binom{n}{2} r \lambda_2(G) \quad \Rightarrow \quad \lambda_2(G) \geq \frac{2}{r(n-1)}$