2025-2-12 Cheeger's Inequality Conductance $\phi(S) = \dfrac{w(\partial(S))}{\min(d(S), d(V-S))}$

$\phi_G = \min_S \phi(S)$. Normalized Laplacian $N \stackrel{\text{def}}{=} D^{-1/2} L D^{-1/2}$ eigvals $0 = \nu_1 < \nu_2 < \dots < \nu_n$

Mentioned $\nu_2/2 \leq \phi_G$. Cheeger: $\phi_G \leq \sqrt{2\nu_2}$. Can find S s.t. $\phi(S) \leq \sqrt{2\nu_2}$

Recall $\nu_2 = \min_{y^T d = 0} \dfrac{y^T L y}{y^T D y}$. Will use $S_t = \{a : y(a) \leq t\}$, some number t

Thm If $y^T d = 0$ and $\rho = \dfrac{y^T L y}{y^T D y}$, $\exists\, t$ s.t. $\phi(S_t) \leq \sqrt{2\rho}$

Replace $y^T d = 0$ with y centered wrt d if $\sum_{a : y(a) > 0} d(a) \leq \dfrac{d(V)}{2}$, $\sum_{a : y(a) < 0} d(a) \leq \dfrac{d(V)}{2}$

To center y, order vertices so $y(1) \leq y(2) \leq \dots \leq y(n)$

choose c to be least integer so that $\sum_{a \leq c} y(a) \geq \dfrac{d(V)}{2}$. $z = y - c\mathbb{1}$

Claim $z^T L z = y^T L y$, and $z^T D z \geq y^T D y$, so $\dfrac{z^T L z}{z^T D z} \leq \dfrac{y^T L y}{y^T D y}$

proof $z^T D z = (y - c\mathbb{1})^T D (y - c\mathbb{1}) = y^T D y + c^2 \mathbb{1}^T D \mathbb{1} - 2c\, \underbrace{\mathbb{1}^T D y}_{= d^T y = 0} \geq y^T D y$

Scale so $z(1)^2 + z(n)^2 = 1$. Will sample t at random. with density $2|t|$.

Thm1 $\mathbb{E}\, w(\partial(S_t)) \leq \sqrt{2\rho}\, \mathbb{E} \min(d(S_t), d(V - S_t))$ $S_t = \{a : z(a) < t\}$,

So, $\exists\, t$ for which holds. Can find it by scanning through t.

Lem1 $\mathbb{E}_t [\min(d(S_t), d(V - S_t))] = z^T D z$

Lem2 $\mathbb{E}\, w(\partial(S_t)) \leq \sum_{(a,b) \in E} w_{a,b} |z(b) - z(a)| (|z(b)| + |z(a)|)$

Choose t in $[z(1), z(n)]$ with density $2|t|$. $\Pr[t \in [a,b]] = \int_{t=a}^{b} 2|t|\, dt$

$$\int_{t=z(1)}^{z(n)} 2|t| = \int_{t=z(1)}^{0} 2|t| + \int_{0}^{z(n)} 2|t| = z(1)^2 + z(n)^2 = 1$$

$$\int_{t=a}^{b} 2|t|\,dt = \mathrm{sgn}(b)\, b^2 - \mathrm{sgn}(a)\, a^2 \qquad \mathrm{sgn}(x) = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$$

z centered wrt d $\Rightarrow$ $\min(d(S_t), d(V-S_t)) = \begin{cases} d(S_t) & \text{if } t \le 0 \\ d(V-S_t) & \text{if } t \ge 0 \end{cases}$

<u>proof of lemma 1</u> $\mathbb{E}_t \min(d(S_t), d(V-S_t)) = \sum_a \Pr[t \le 0 \text{ and } a \in S_t] + \Pr[t>0 \text{ and } a \in V-S_t]$

(recall $z(c)=0$) $= \sum_{a<c} \Pr[z(a)<t \text{ and } t<0]\, d(a) + \sum_{a \ge c} \Pr[z(a) \ge t \text{ and } t \ge 0]$

$= \sum_{a<c} \Pr[z(a)<t<0]\, d(a) + \sum_{a\ge c} \Pr[0 \le t \le z(a)]\, d(a) = \sum_{a<c} z(a)^2 d(a) + \sum_{a \ge 0} z(a)^2 d(a) = \sum_a z(a)^2 d(a)$

<u>proof of lem 2</u> $\mathbb{E}\, w(\partial(S_t)) = \sum_{(a,b)\in E} w(a,b) \Pr[a<t\le b]$

$\Pr[a<t\le b] = \mathrm{sgn}(z(b))\, z(b)^2 - \mathrm{sgn}(z(a))\, z(a)^2$

when $\mathrm{sgn}(a)=\mathrm{sgn}(b)$ $= |z(b)^2 - z(a)^2| = |z(b)-z(a)|\,(|z(b)|+|z(a)|)$

when $\mathrm{sgn}(b) \neq \mathrm{sgn}(a)$ $= z(b)^2 + z(a)^2 \le (z(b)-z(a))^2$

$= (z(b)-z(a))\,(|z(b)|+|z(a)|)$

<u>proof of Thm 1</u> $\sum_{(a,b)\in E} w_{a,b}\, |z(b)-z(a)|\,(|z(b)|+|z(a)|) \le$

$$\left(\sum_{a\sim b} w_{a,b}(z(b)-z(a))^2\right)^{1/2} \left(\sum_{a\sim b} w_{a,b}(|z(b)|+|z(a)|)^2\right)^{1/2}$$

$z^T L z \le \rho\, z^T D z$ $\qquad \le 2\sum_{a\sim b} w_{a,b}\left(|z(b)|^2 + |z(a)|^2\right) = 2\sum_a d(a)\, z(a)^2 = 2 z^T D z$

So, $\mathbb{E}\, w(\partial(S_t)) \le \sqrt{\rho \cdot z^T D z}\,\sqrt{2 z^T D z} = \sqrt{2\rho}\; z^T D z = \sqrt{2\rho}\; \mathbb{E} \min(d(S_t), d(V-S_t))$
