CPSC 462/562 - Spectral Graph Theory (Spring 2025)	**Note 1**

Eigenvalues and Eigenvectors of Abelian Cayley Graphs

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In this lecture, we study a family of graphs arising from group theory with a very clean characterization for eigenvalues and eigenvectors, namely Cayley graphs. We will see that they generalize some of the graphs in the zoo we have encountered in previous lectures, placing Cayley graphs as the main exhibit in the zoo. This note is based on Chapter 7 of [1] and Chapter 16 of [2].

1 Elements of Group Theory

Before defining the graphs we are interested in and analyzing their spectrum, we need to recall a few definition and facts from group theory. If you have taken an introductory abstract algebra class, you can safely skip to the next section.

Definition 1 (Abelian Group)**.** *A pair $(\Gamma, \circ)$ composed of a set Γ and an operation $\circ : \Gamma \times \Gamma \to \Gamma$ is called a group if it satisfies the following requirements:*

1. ***Associativity**: For all $a, b, c \in \Gamma$ we have $(a \circ b) \circ c = a \circ (b \circ c)$*
2. ***Identity element**: There exists $e \in \Gamma$ such that for all $a \in \Gamma$, we have $a \circ e = e \circ a = a$*
3. ***Inverse**: For all $a \in \Gamma$ there exists $b \in \Gamma$ such that $a \circ b = b \circ a = e$*

The group is called Abelian if it also satisfies:

4. ***Commutativity**: For all $a, b \in \Gamma$, we have $a \circ b = b \circ a$*

All groups considered in the subsequent sections of this lecture are Abelian, finite ($|\Gamma| < \infty$), and additive ($\circ = +$), so we will often denote them using just the underlying set Γ. In particular, we denote the additive identity as $e = 0$ and the additive inverse of a as $-a$. The main example of a finite Abelian group is the cyclic group with n elements $\Gamma = \mathbb{Z}/n\mathbb{Z}$, equipped with the operation of addition modulo n, also denoted by $\{0, 1, ..., n-1\} \mod n$.

In Sections 2 and 3 we will discuss graphs based on Abelian groups. In Section 4 we will present a general theory for their spectrum which requires the notion of group homomorphism.

Definition 2 (Group homomorphism)**.** *Given two groups $(\Gamma, \circ)$ and $(\Phi, \cdot)$, a group homomorphism from $(\Gamma, \circ)$ to $(\Phi, \cdot)$ is a function $f : \Gamma \to \Phi$ that is compatible with the group structure, i.e. for all $a, b \in \Gamma$,*

$$f(a \circ b) = f(a) \cdot f(b)$$

When the reference group is the multiplicative complex group $(\mathbb{C} \setminus \{0\}, \cdot)$, homomorphisms have a special name.

Definition 3 (Characters)**.** *Let $(\Gamma, \circ)$ be a finite group, then the function $\chi : \Gamma \to \mathbb{C}$ is a character of $(\Gamma, \circ)$ if it is a homomorphism from $(\Gamma, \circ)$ to $(\mathbb{C} \setminus \{0\}, \cdot)$.*

Since we are working with finite groups, a character is just a complex-valued vector defined on the group. Note that the all-1 vector is a character of any group, by definition.

Recall the complex inner product between two $|\Gamma|$-dimensional vectors $x, y \in \mathbb{C}^\Gamma$

$$\langle x, y \rangle = \sum_{a \in \Gamma} x(a) y^*(a)$$

where $y^*(a) = \mathrm{Re}(y(a)) - i\mathrm{Im}(y(a))$ is the complex conjugate. We have the following Lemma about different characters of the same group.

Lemma 4. *Let Γ be a finite group and let χ be a character of Γ with at least one entry different from 1. Then*

$$\langle \chi, \mathbf{1} \rangle = 0$$

More generally, if $\chi_1 \neq \chi_2$ are two different characters of Γ, then

$$\langle \chi_1, \chi_2 \rangle = 0$$

Proof. For the first claim, let b be such that $\chi(b) \neq 1$, and note that $\pi(a) = a + b$ is a permutation of Γ. Then,

$$\chi(b) \sum_{a \in \Gamma} \chi(a) = \sum_{a \in \Gamma} \chi(b + a) = \sum_{a \in \Gamma} \chi(a)$$

This implies that

$$(\chi(b) - 1) \sum_{a \in \Gamma} \chi(a) = 0$$

Since $\chi(b) \neq 1$, the first claim holds $\langle \chi, \mathbf{1} \rangle = \sum_{a \in \Gamma} \chi(a) = 0$.

For the second part, we need to show that characters form a multiplicative sub-group of $\mathbb{C}^\Gamma$. In particular, it is easy to check that if χ is a character, then χ^* is also a character, and $\chi^* = \chi^{-1}$. Moreover, the function $\chi = \chi_1 \cdot \chi_2^*$ is also a character, which just follows from associative properties of (pointwise) scalar multiplication.

Let b be such that $\chi_1(b) \neq \chi_2(b)$, then $\chi(b) \neq 1$, and we can apply the first part to observe that $\sum_{a \in \Gamma} \chi(a) = 0$. Expanding,

$$\sum_{a \in \Gamma} \chi(a) = \sum_{a \in \Gamma} \chi_1(a) \chi_2(a)^* = \langle \chi_1, \chi_2 \rangle = 0$$

□

Since characters are orthogonal, there are at most $|\Gamma|$ of them. For Abelian groups, this bound is tight.

Lemma 5. *Every finite Abelian group Γ has exactly $|\Gamma|$ characters.*

We defer the proof of this fact to Section 4, and it uses the fact that every finite Abelian group is isomorphic to a product of cyclic groups, so it boils down to producing $|\Gamma|$ distinct characters for the cyclic group $\mathbb{Z}/n$ and understanding how characters behave under group products.

So far, we showed that every finite Abelian group with n elements has n orthogonal characters, including the all-1 vector. It is natural to wonder whether there is some graph defined on the group that has Laplacian eigenvectors equal to the characters, and what the eigenvalues would look like. In the next section, we present a family of graphs with exactly this property, and rediscover some familiar faces.

2 Cayley Graphs

A Cayley graph is a graph whose vertices are the elements of a group Γ, and whose edges are specified by a set of *generators* $S \subseteq \Gamma$. Two vertices a and b are connected by an edge if there is an $s \in S$ such that $a + s = b$.[1] To make the graph undirected, we require S to be closed under inversion, i.e. $s \in S \implies s^{-1} \in S$, which implies $a + s = b \iff b + (-s) = a$. We denote this Cayley graph by $Cay(\Gamma, S)$.

Note that all Cayley graphs are unweighted and $|S|$-regular. We have already encountered two Cayley graphs in the zoo of graphs.

[1] We use additive notation because we focus on Cayley graphs arising from Abelian groups.

Example 6 (Hypercube)**.** *The d-dimensional hypercube is the Cayley graph defined on the group* $\Gamma = (\mathbb{Z}/2\mathbb{Z})^d$ *with generators*

$$S = \{(1, 0, ..., 0), (0, 1, , ..., 0), ..., (0, 0, ..., 1)\}$$

Note that S is closed under inversion because $s + s = 0$*, so every element in S is its own inverse. In other words,* $H_d = Cay((\mathbb{Z}/2\mathbb{Z})^d, S)$*.*

Example 7 (Ring graph)**.** *The n-vertex ring graph (or cycle) is the Cayley graph defined on the group* $\Gamma = \mathbb{Z}/n\mathbb{Z}$ *with generators* $S = \{-1, 1\}$*. In other words,* $R_n = Cay(\mathbb{Z}/n\mathbb{Z}, S)$*.*

The most remarkable property of Cayley graphs of Abelian groups is that their eigenvectors only depend on the group, and not on the edge structure. The eigenvalues depend on the generator (hence on the edge structure), but they are easy to find once we know the eigenvectors.

In the next section we will show that the eigenvectors of the hypercube are also eigenvectors of generalized hypercubes, i.e. Abelian Cayley graphs on the same group with a different set of generators. This will give a simple formula for the eigenvalues. However, the reader might find the fact that all these eigenvectors are just "checked" a little unsatisfactory, and it might not be clear a priori what they should look like. In the subsequent section, we will provide a more general theory for the spectrum of Abelian Cayley graphs that unifies all these examples and shows that computing the eigenvectors amounts to computing the characters of the group.

3 Warm-up: Generalized Hypercubes and their Spectrum

Consider the group $\Gamma = (\mathbb{Z}/2\mathbb{Z})^d$ of binary vectors of length d, which can also be written as $\{0, 1\}^d \mod 2$, and let $S \subseteq \Gamma$ have size $|S| = k$. Note that any S is a valid set of generators, because $s + s = 0$ for any binary vector. We are interested in the Cayley graphs $G = (V, E)$ with vertex set $V = \Gamma$ and generator set S. More concretely, the edge set is

$$E = \{(a, a + s_i) : a \in V, 1 \leq i \leq k\}$$

For any $x \in \{0, 1\}^d$, we can define the function (vector) ψ_x from V to the reals as

$$\psi_x(a) = (-1)^{x^\top a} \tag{1}$$

Note that we could have written $x \in \Gamma$, but we chose not to do it because you should not think of these indices as vertices of the graph. In fact, the ψ_x are Fourier coefficients for the group Γ, which form an orthogonal basis of the space of linear functions defined on the group. Since we are working in finite dimensions, this space is isomorphic to the group itself, but identifying group elements with Fourier elements is the wrong perspective.

Recall that the set of vectors defined in (1) are the eigenvectors of the hypercube. Notice that they are also characters of the group Γ, because

$$\psi_x(a + b) = (-1)^{x^\top (a+b)} = (-1)^{x^\top a}(-1)^{x^\top b} = \psi_x(a)\psi_x(b)$$

We can now verify that these vectors are indeed the eigenvectors of the Laplacian matrix of generalized hypercubes $Cay((\mathbb{Z}/n\mathbb{Z}), S)$.[2]

Theorem 8. *For each* $x \in \{0, 1\}^d$*, the vector* ψ_x *is an eigenvector of the Laplacian matrix of G with eigenvalue*

$$k - \sum_{i=1}^{k} (-1)^{x^\top s_i}$$

[2]Since all Cayley graphs are regular, we could equivalently consider the adjacency matrix.

Proof. Let $L = kI - M$ be the Laplacian of the graph, then the result simply follows from checking the eigenvector equation for any vertex $a \in V$,

$$\begin{aligned}
(L\psi_x)(a) &= k\psi_x(a) - \sum_{b\sim a} \psi_x(b) \\
&= k\psi_x(a) - \sum_{i=1}^{k} \psi_x(a+s_i) \\
&= k\psi_x(a) - \sum_{i=1}^{k} \psi_x(a)\psi_x(s_i) \\
&= \left(k - \sum_{i=1}^{k} \psi_x(s_i)\right)\psi_x(a) \\
&= \left(k - \sum_{i=1}^{k} (-1)^{x^\top s_i}\right)\psi_x(a)
\end{aligned}$$

□

4 Spectrum of Abelian Cayley Graphs

We saw that the eigenvectors of the generalized hypercube are the characters of the underlying group, no matter which set of generators are used. In this section we show that this is not a special property of the hypercube, and it holds for all Abelian Cayley graphs.

Theorem 9. *Let Γ be a finite Abelian group and χ be a character of Γ. Then for any set of generators $S = \{s_1, ..., s_k\} \subseteq \Gamma$, the vector χ is an eigenvector of the Laplacian of the $Cay(\Gamma, S)$ with eigenvalue*

$$\lambda_\chi = k - \sum_{i=1}^{k} \chi(s_i)$$

Proof. The proof is actually identical to the proof of Theorem 8, because we only used the fact that the given vector is a character and the definition of neighboring vertices for Cayley graphs.

$$\begin{aligned}
(L\chi)(a) &= k\chi(a) - \sum_{b\sim a} \chi(b) \\
&= k\chi(a) - \sum_{i=1}^{k} \chi(a+s_i) \\
&= k\chi(a) - \sum_{i=1}^{k} \chi(a)\chi(s_i) \\
&= \left(k - \sum_{i=1}^{k} \chi(s_i)\right)\chi(a) \\
&= \lambda_\chi \chi(a)
\end{aligned}$$

□

We showed that characters satisfy the eigenvector equations for the Laplacian, but we cannot conclude that they are eigenvectors yet, because they are complex-valued. However, due to the symmetry of the generator set S, we will show how to construct real eigenvectors from characters. We start by describing the characters of the cyclic group.

Lemma 10. *Consider the cyclic group $\Gamma = \mathbb{Z}/n\mathbb{Z}$. Then for any $r \in \{0, ..., n-1\}$, the vector*

$$\chi_r(a) = e^{\frac{2\pi i r a}{n}}$$

is a character of Γ. In particular, Γ has n distinct characters.

Proof. We need to show that each χ_r is a homomorphism from $\mathbb{Z}/n\mathbb{Z}$ to the complex multiplicative group. Note that $\chi(0) = 1$, and $\chi(a) = \chi(-a)^{-1}$, so identity and inverses are preserved. Moreover, for any $a, b \in \mathbb{Z}/n\mathbb{Z}$,

$$\chi_r(a+b) = e^{\frac{2\pi i r(a+b)}{n}} = e^{\frac{2\pi i r a}{n}} e^{\frac{2\pi i r b}{n}} = \chi_r(a)\chi_r(b)$$

This shows that every χ_r is a character. Since $\chi_r(1)$ is different for each r, all these characters are distinct. The second claim now follows from Lemma 4, since there can be at most n characters. □

The following lemma shows that characters behave as expected under group products.

Lemma 11. *If χ_1 is a character of Γ_1 and χ_2 is a character of Γ_2, then $\chi_1 \cdot \chi_2$ is a character of $\Gamma_1 \times \Gamma_2$. In particular, if $\Gamma_1 = \mathbb{Z}/n_1\mathbb{Z}$ and $\Gamma_1 = \mathbb{Z}/n_2\mathbb{Z}$, then $\Gamma_1 \times \Gamma_2$ has $n_1 n_2$ characters.*

Proof. The second part of the claim follows immediately from the first part and Lemma 10, see [2, Theorem 16.9] for the proof of the first part. □

The proof of Lemma 5 follows by these Lemmas 10 and 11, since we can use them to enumerate $|\Gamma|$ distinct characters for any finite Abelian group Γ.

Theorem 12. *Let Γ be a finite Abelian group and χ be a character of Γ such that $\chi \neq \chi^{-1}$. Then χ^{-1} is also a character of Γ, and for any set of generators $S \subseteq \Gamma$, the vectors $v_+ = \chi + \chi^{-1}$ and $v_- = \chi - \chi^{-1}$ are real eigenvectors of the Laplacian of $Cay(\Gamma, S)$ with (real) eigenvalue λ_χ.*

Proof. We start by showing that χ^{-1} is a character of Γ. For any $a, b \in \Gamma$,

$$\chi^{-1}(a+b) = (\chi(a+b))^{-1} = (\chi(a)\chi(b))^{-1} = \chi^{-1}(a)\chi^{-1}(b)$$

We can partition the set of generators $S = S_1 \cup (-S_1)$ into a disjoint union of sets containing "half" the generators and their inverses, respectively.[3] By Theorem 9, it is an eigenvector with eigenvalue

$$\begin{aligned} \lambda_{\chi^{-1}} &= k - \sum_{s \in S} \chi^{-1}(s) = k - \sum_{s \in S_1} \chi^{-1}(s) + \sum_{s \in -S_1} \chi^{-1}(s) \\ &= k - \sum_{s \in S_1} (\chi^{-1}(s) + \chi^{-1}(-s)) \\ &= k - \sum_{s \in S_1} (\chi(-s) + \chi(s)) \\ &= k - \sum_{s \in S} \chi(s) = \lambda_\chi \end{aligned}$$

This shows that both v_+ and v_- are eigenvectors of L with eigenvalue λ_χ. To show that they have real entries, we use the explicit formula for characters of Abelian groups. For simplicity, consider the case $\Gamma = \mathbb{Z}/n\mathbb{Z}$, since the product case can be handled similarly. By Lemma 10, there is some $r \in \{0, ..., n-1\}$ such that for any $a \in \Gamma$, we have

$$\begin{aligned} v_+(a) &= \chi(a) + \chi^{-1}(a) = e^{\frac{2\pi i r a}{n}} + e^{-\frac{2\pi i r a}{n}} \\ &= \cos(2\pi ra/n) + i\sin(2\pi ra/n) + \cos(2\pi ra/n) - i\sin(2\pi ra/n) \\ &= 2\cos(2\pi ra/n) \end{aligned}$$

Similarly, we can show that $v_-(a) = 2\sin(2\pi ra/n)$.

Showing that λ_χ is real uses very similar calculations, so we leave it as an exercise to the reader. □

[3] The set S_1 does not necessarily have half the generators, for example the hypercube has $S_1 = S$ and $-S_1 = S_1$. The corner case is when one generator is its own inverse, but in that case $\chi^{-1}(s) = \chi(-s) = \chi(s)$, so we should really consider a disjoint tri-partition of the generator set.

For every pair of complex characters χ, χ^{-1} such that $\chi \neq \chi^{-1}$, this theorem gives us two orthogonal eigenvectors, since $v_+^\top v_- = \langle \chi, \chi \rangle - \langle \chi^{-1}, \chi^{-1} \rangle = n - n = 0$. As a consequence, we showed that every eigenvalue of Abelian Cayley graphs corresponding to characters that are not their own inverse must have multiplicity at least 2. The eigenvectors produced by different characters are clearly orthogonal, by Lemma 4, so there are as many orthogonal eigenvectors as characters. By Lemma 5, there are n characters, so Theorems 9 and 12 describe all the possible Laplacian eigenvectors.

As a sanity check, note that the eigenvectors of Cayley graphs on $\mathbb{Z}/n\mathbb{Z}$ are exactly the eigenvectors of the ring graph studied in the previous lecture.

$$x_k(a) = \cos(2\pi ka/n) \quad \text{for } 0 \leq k < n/2$$
$$y_k(a) = \sin(2\pi ka/n) \quad \text{for } 0 < k \leq n/2$$

5 Further reading and Applications

Aside from having a very elegant theory for their Laplacian eigenvalues and eigenvectors, Cayley graphs are extremely useful in computer science applications. To list a few, they can be used to construct explicit expanders (although for sparse expanders we need non-Abelian Cayley graphs), design both classical and quantum error correcting codes, and prove lower bounds for graph problems.

We refer the reader to [1, Chapter 7.6] for a more detailed discussion on Cayley graphs and expansion and further references.

References

[1] Daniel Spielman. Spectral and algebraic graph theory. *Yale lecture notes, draft of December*, 4:47, 2019.

[2] Luca Trevisan. Lecture notes on graph partitioning, expanders and spectral methods. *University of California, Berkeley, https://people. eecs. berkeley. edu/luca/books/expanders-2016. pdf*, 2017.