Conductance $\phi(S) = \dfrac{w(\partial(S))}{\min(d(S), d(V-S))}$

for a weighted graph, $w(F) =$ sum of edge wts in F

$d(S) = \sum_{a \in S} d(a)$ wted degree.

Recall Normalized Laplacian $N = D^{-1/2} L D^{-1/2}$

eigenvalues $0 = \nu_1 \le \nu_2 \le \cdots \le \nu_n$

Saw $\phi(S) \ge \nu_2/2 \quad \forall S$

Cheeger's Inequality: $\exists S$ s.t. $\phi(S) \le \sqrt{2\nu_2}$

Recall $\nu_2 = \min_{y^T d = 0} \dfrac{y^T L y}{y^T D y}$

Def: $S_t = \{a : y(a) \le t\}$

Cheeger: For any vector y with $\boxed{y^T d = 0}$ y balanced wrt d

and $\dfrac{y^T L y}{y^T D y} = \rho$, $\exists t$ s.t. has $\phi(S_t) \le \sqrt{2\rho}$

So, does not need to be an eigenvector

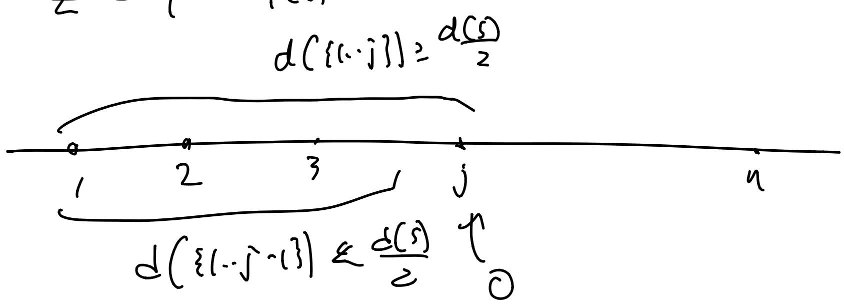
Def y is balanced wrt d if $\sum_{a: y(a) > 0} d(a) \le \frac{d(V)}{2}$

and $\sum_{a: y(a) < 0} d(a) \le \frac{d(V)}{2}$

Renumber vertices s.t. $y(1) \le y(2) \le \dots \le y(n)$

Choose j to be least integer s.t. $\sum_{a \le j} d(a) \ge \frac{d(V)}{2}$

Set $z = y - y(j)\mathbf{1}$. z is balanced.



Claim Let $z_s = y - s\mathbf{1}$ for $d^T y = 0$

$z_s^T D z_s$ is minimized at $s = 0$.

$$\frac{d}{ds} z_s^T D z_s = \frac{d}{ds} \sum_a d(a)(y(a) - s)^2 = 2 d^T y$$

So, $\frac{z^T L z}{z^T D z} \le \frac{y^T L y}{y^T D y}$

Normalize: Assume wolog $z(1)^2 + z(n)^2 = 1$

Will sample t at random, and prove

$$\mathop{\mathbb{E}}_t \, w(\partial(S_t)) \le \sqrt{2\rho} \, \mathop{\mathbb{E}}_t [\min(d(S_t), d(V - S_t))] \quad (*)$$

So, $\exists$ a t for which this holds.

Note: can find one in linear time, given z.

Choose t in $[z(1), z(n)]$ with density $2|t|$.

That is $\Pr[t \in [a,b]] = \int_{t=a}^{b} 2|t| \, dt$.

$$\int_{t=z(1)}^{z(n)} 2|t| = \int_{t=z(1)}^{0} 2|t| + \int_{t=0}^{z(n)} 2|t| = z(1)^2 + z(n)^2 = 1$$

In general, $\int_{t=a}^{b} 2|t| \, dt = \mathrm{sgn}(b)\, b^2 - \mathrm{sgn}(a)\, a^2$

z centered wrt d $\Rightarrow$

$t \le 0 \Rightarrow \min(d(S_t), d(V - S_t)) = d(S_t)$

$t \ge 0 \Rightarrow$ " $= d(V - S_t)$

Lem 1 $\mathbb{E}_t\left[\min\left(d(S_t), d(U-S_t)\right)\right] = z^T D z$

proof

$$\mathbb{E}_t\, d(S_t) = \sum_a \Pr[a \in S_t]\, d(a) = \sum_a \Pr[z(a) \le t]\, d(a)$$

$z(j) = 0$ so.

$$\mathbb{E}\left[\min\left(d(S_t), d(U-S_t)\right)\right]$$

$$= \sum_{a<j} \Pr[z(a) < t \text{ and } t < 0]\, d(a) + \sum_{a \ge j} \Pr[z(a) \ge t \text{ and } t > 0]\, d(a)$$

$$= \sum_{a<j} \Pr[z(a) < t < 0]\, d(a) + \sum_{a \ge j} \Pr[0 < t < z(a)]\, d(a)$$

$$= \sum_{a<j} z(a)^2 d_a + \sum_{a \ge j} z(a)^2 d(a) = \sum_a d(a)\, z(a)^2 = z^T D z$$

Lem 2 $\mathbb{E}\, w(\partial(S_t)) \le \sum_{a\sim b} w_{a,b} |z(a) - z(b)| \cdot (|z(a)| + |z(b)|)$

proof $= \sum_{a\sim b} w_{a,b} \Pr[z(a) < t \le z(b)]$

$= \operatorname{sgn}(z(b)) \cdot z(b)^2 - \operatorname{sgn}(z(a))\, z(a)^2$

when $\operatorname{sgn}(b) \ne \operatorname{sgn}(a)$

$z(b)^2 + z(a)^2 \le (z(b) - z(a))^2 = |z(a) - z(b)| (|z(a)| + |z(b)|)$

when $\operatorname{sgn}(z(b)) = \operatorname{sgn}(z(a))$ $= |z(a)^2 - z(b)^2|$

$= |(z(a) - z(b))(z(a) + z(b))| \le |z(a) - z(b)| \cdot (|z(a)| + |z(b)|)$

To prove (*), use

$$\sum_{a\sim b} w_{a,b} |z(a) - z(b)| \cdot (|z(a)| + |z(b)|)$$

$$\le \underbrace{\sqrt{\sum_{a\sim b} w_{a,b}(z(a) - z(b))^2}}_{= z^T L z \,\le\, \rho z^T D z} \sqrt{\sum_{a\sim b} w_{a,b} (|z(a)| + |z(b)|)^2}$$

$$\le \sum_{a\sim b} w_{a,b} \cdot 2 (z(a)^2 + z(b)^2) = 2 \sum_a d(a)\, z(a)^2 = 2 z^T D z$$

So, get $\ge \sqrt{\rho z^T D z}\sqrt{2 z^T D z} = \sqrt{2\rho}\, z^T D z$

$\Rightarrow \mathbb{E}\, w(\partial(S_t)) \le \sqrt{2\rho}\, z^T D z = \sqrt{2\rho}\, \mathbb{E} \min(d(S_t), d(V - S_t))$