Erdős-Renyi. Each edge prob p

$$\mu_1 \approx pn \qquad |\mu_i| \le (1+o(1))\, 2\sqrt{np(p-1)} \qquad i \ge 2,$$

$$R(a,b) = \begin{cases} 1-p & \text{prob } p \\ -p & \text{prob } 1-p \end{cases} \qquad \mathbb{E}\, R(a,b) = 0$$

$$M = pJ - pI + pR, \qquad J = \text{all 1s matrix}$$

$$\mathbb{E} M = p(J - I)$$

Let $\rho_1 \dots \rho_n$ be eigvals of R.

$$\sum \rho_i^{\ell} = \mathrm{Tr}(R^{\ell})$$

Care about $\|R\| = \max_i |\rho_i|$

$$\|R\|^{\ell} \le \mathrm{Tr}(R^{\ell})$$

$$\|R\| \le \left(\mathrm{Tr}(R^{\ell})\right)^{1/\ell}$$

Will prove if $np(1-p) \ge 2 \cdot \ell^8$

$$\mathbb{E}\,\mathrm{Tr}(R^{\ell}) \le 2n\left(2\sqrt{np(1-p)}\right)^{\ell} \triangleq u^{\ell} \qquad u = (2n)^{1/\ell}\, 2\sqrt{np(1-p)}$$

So, for $\varepsilon > 0$

$$\Pr\left[\|R\| \ge (1+\varepsilon)u\right] \le \Pr\left[\mathrm{Tr}(R^{\ell}) \ge (1+\varepsilon)^{\ell} u^{\ell}\right]$$

$$\le \Pr\left[\mathrm{Tr}(R^{\ell}) \ge (1+\varepsilon)^{\ell}\, \mathbb{E}\,\mathrm{Tr}(R^{\ell})\right]$$

$$\leq \frac{1}{(1+\varepsilon)\ell} \approx e^{-\varepsilon \ell}$$

so small if $\varepsilon < 1/\ell$

$$(2n)^{1/\ell} = \exp\left(\ln(2n)/\ell\right) \approx 1 + \frac{\ln(2n)}{\ell} \quad \text{so close to } 1$$

$$R^{\ell}(a_0, a_0) = \sum_{a_1} R(a_0, a_1)\, R^{\ell-1}(a_1, a_0)$$

$$= \sum_{a_1, \dots, a_{\ell-1}} R(a_{\ell-1}, a_0) \cdot \prod_{i=1}^{\ell-1} R(a_{i-1}, a_i)$$

$$\mathbb{E}\, R^{\ell}(a_0, a_0) = \sum_{a_1 \dots a_{\ell-1}} \mathbb{E}\, R(a_{\ell-1}, a_0) \prod_{i=1}^{\ell-1} R(a_{i-1}, a_i) \quad (*)$$

for indep X, Y $\mathbb{E}[XY] = \mathbb{E}X \cdot \mathbb{E}Y$

and $\mathbb{E}\, R(a, b) = 0$.

So, only nonzero when each term appears at least twice

Let $\{b_j, c_j\}_j$ be the distinct pairs in $\{a_{\ell-1}, a_0\} \cup \{a_{i-1}, a_i\}_{i=1}^{\ell-1}$

and let $\{b_j, c_j\}$ appear d_j times.

Then $(*) = \prod_j \mathbb{E}\, R(b_j, c_j)^{d_j}$

$$\mathbb{E}\, R(b_j, c_j)^{d_j} = p(1-p)\left[(1-p)^{d_j - 1} - (-p)^{d_j - 1}\right] \leq p(1-p) \quad d \geq 2$$

$\mathbb{E}\, R^{\ell}(a_0, a_0) =$ sum over seq $a_1 \ldots a_{\ell-1}$

s.t. each pair occurs ≥ 2 times

$\cdot\ (p(1-p))^{\#\text{ distinct pairs}}$.

Say $a_0, a_1, \ldots a_\ell$ closed walk of length ℓ if $a_0 = a_\ell$

is significant if each pair $\{a_{i-1}, a_i\}$ occurs at least twice

$W_{n,\ell,k}$ = # sig closed walks length ℓ with

k distinct elements among $a_1, \ldots, a_{\ell-1}$

$$\mathbb{E}\, \mathrm{Tr}(R^\ell) \leq \sum_{k=1}^{\ell/2} W_{n,\ell,k} \left(p(1-p)\right)^k$$

distinct pairs $\geq$ # distinct elements

Will prove $W_{n,\ell,k} \leq n^{k+1}\, 2^\ell\, \ell^{4(\ell-2k)}$

Thm If ℓ even and $np(1-p) \geq 2\ell^8$, then

$$\mathbb{E}\, \mathrm{Tr}(R^\ell) \leq 2n \left(2\sqrt{np(1-p)}\right)^\ell.$$

proof let $t_k = n^{k+1}\, 2^\ell\, \ell^{4(\ell-2k)} \left(p(1-p)\right)^k$

Then $t_k \geq 2 t_{k-1}$ for $np(1-p) \geq 2\ell^8$

so $$\sum_{k=1}^{\ell/2} W_{n,\ell,k} \left(p(1-p)\right)^k \leq \sum_{k=1}^{\ell/2} t_k \leq 2 t_{\ell/2} = (2n)\, 2^\ell\, n^{\ell/2} \left(p(1-p)\right)^{\ell/2}$$

$$= 2n \left(2\sqrt{np(1-p)}\right)^\ell$$

Bound on $W_{n,l,k}$ $\qquad a_0, a_1, \dots, a_{l-1}, a_0$

$S = \{ i : a_i \neq a_j \text{ for } j < i \}$ "appears first" $\qquad \binom{l-1}{k}$

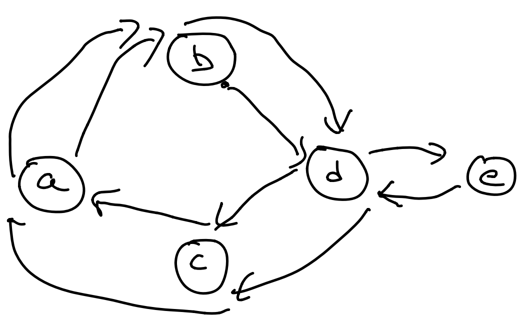
For each $i \in S$, record a_i. $\sigma(i) = a_i$ $\qquad n^k$

And, record a_0 $\qquad n$

For $i \notin S$, record $j \in S$ s.t. $a_i = a_j$ or 0. $\tau : [l-1] \setminus S \to S$ $\qquad (k+1)^{l-1-k}$

So, $W_{n,l,k} \leq \binom{l-1}{k} n^{k+1} (k+1)^{l-1-k} \leq n^{k+1} 2^{l-1} (k+1)^{l-1-k}$

Example



$k = 4$ $\quad S = \{1, 2, 3, 6\}$

$a_0 = b$ $\quad \sigma(1) = d$ $\quad \tau(4) = 0$

$\sigma(2) = c$ $\quad \tau(5) = 1$

$\sigma(3) = a$ $\quad \tau(7) = 1$

$\sigma(6) = e$ $\quad \tau(8) = 2$

$\tau(9) = 3$

step	0	1	2	3	4	5	6	7	8	9	10
vertex	b	a	d	c	a	b	e	b	d	c	b

$W_{n,l,k} \leq n^{k+1} 2^{l-1} (k+1)^{l-1-k}$ gets us to $c\sqrt{np(1-p)}\lg n$

Can improve for big k.

When $k = \ell/2$, do not need τ.

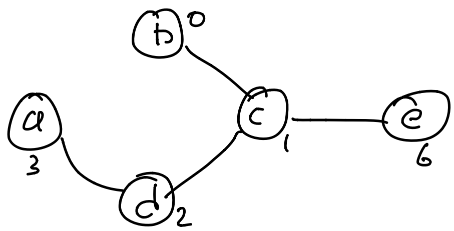
no choice in a_i for $i \notin S$.

Call pair (a_{i-1}, a_i) for $i \in S$ a <u>tree</u> edge.

Because they form a tree. Are $k = \ell/2$ $i \in S$.

each has an edge. So, k edges, $k+1$ vertices (with a_0)

and connected (by induction)

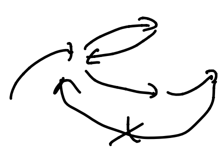


Must use each tree edge twice, and no others

Look at tree edges used exactly once through step $i-1$.

If $i \notin S$, will show is exactly one touching a_{i-1}, so must follow it.

Because only use tree edges, and $i \notin S$ so must be used before

Why only one? 

Graph G_i = Tree edges used exactly once in $a_0, \dots, a_i$

vertices attached to them, and a_i

v_i = # vertices in G_i, e_i = # edges

$v_0 = 1$ $e_0 = 0$ $v_\ell = 1$ $e_\ell = 0$

For $i \in S$ $\quad v_i = v_{i-1} + 1 \quad e_i = e_{i-1} + 1$

$i \notin S$, if $\deg_{G_{i-1}}(a_{i-1}) = 1$,

remove (a_{i-1}, a_i) and a_{i-1}, so

$v_i = v_{i-1} - 1 \quad e_i = e_{i-1} - 1$

if $\deg_{G_{i-1}}(a_{i-1}) \geq 2$,

remove (a_{i-1}, a_i) but not a_{i-1}

$v_i = v_{i-1} \quad e_i = e_{i-1} - 1$.

Contradiction.

So, $W_{n, \ell, \ell/2} \leq 2^{\ell-1} n^{\ell/2+1}$

To show $W_{n,\ell,k} \leq 2^{\ell-1} n^{k+1} \ell^{4(\ell-2k)}$

$T = \{ i : \{a_{i-1}, a_i\} \in G_{i-1}, \text{ and } \deg_{G_{i-1}}(a_i) = 1 \}$ like tree

Step i ambiguous if $\{a_{i-1}, a_i\} \in G_i$ but $\deg_{G_{i-1}}(a_i) > 1$

$e_i = e_{i-1} - 1 \quad v_i = v_{i-1}$

Step i <u>extra</u> if $\{a_{i-1}, a_i\}$ not tree edge
or already used ≥ 2 times.

$e_i = e_{i-1} \quad v_i \geq v_{i-1} - 1$

can go down if $\deg_{G_{i-1}}(a_{i-1}) = 0$

For i ambiguous or extra, use τ to record $j \notin S$ s.t. $a_i = a_j$

$\#$ ambiguous $\le \#$ extra $= x$

$\#$ extra $\le \ell - 2k$ steps not in T or S are extra or ambiguous

$\Rightarrow$ spec $S, T, \sigma, \tau, \sigma_0$

$$\binom{\ell-1}{k} \quad \binom{\ell-1-k}{2x} \quad n^k \quad (k+1)^{2x} \quad n$$

$$\le 2^{\ell-1} (\ell-1-k)^{2x} (k+1)^{2x} n^{k+1}$$

$$\le 2^{\ell-1} n^{k+1} (\ell)^{4x}$$

$$= 2^{\ell-1} n^{k+1} \ell^{4(\ell-2k)}$$