Laplacian Spectrum of Graphs

Recall: $(Lx)(a) = \sum_{b\sim a} x(a) - x(b) = d(a)x(a) - \sum_{b\sim a} x(b)$

proved $\theta(S) = \frac{|\partial(S)|}{|S|} \geq \lambda_2\left(1 - \frac{|S|}{n}\right)$

$\lambda_1 = 0$

$\sum_{i=1}^{n} \lambda_i = Tr(L) = \sum_a d(a)$

K_n. $E = \{(a,b) : a \neq b\}$ n vertices.

Has $\lambda_2 = \cdots = \lambda_n = n$

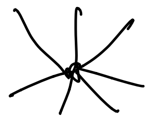
Let ψ be vec s.t. $1^T\psi = 0$

$$(L\psi)(a) = \sum_{b\neq a} \left(\psi(a) - \psi(b)\right) = (n-1)\psi(a) - \underbrace{\sum_{b\neq a} \psi(b)}_{\psi(a)}$$

$$= n\psi(a)$$

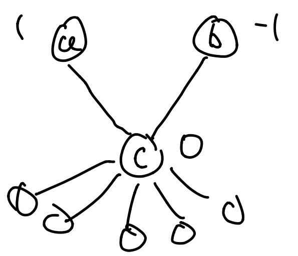
$\theta(S) = \frac{|\partial(S)|}{|S|} = \frac{|S|\cdot|V-S|}{|S|} = |V-S| = n\left(1 - \frac{|S|}{n}\right)$ tight.

Star Graph S_n. $V = \{1..n\}$ $E = \{(1,a) : a \geq 2\}$



Claim $\delta_a - \delta_b$ is eigvec of eigvalue 1.

<u>Lem</u> Let G be a graph with degree-1 vertices a and b s.t. (a,c) and $(b,c) \in E$.
Then $x = \delta_a - \delta_b$ is eigvec of eigval 1.



$x(a) - x(c) = 1 = x(a)$

$x(b) - x(c) = -1 = x(b)$

$2x(c) - x(a) - x(b) = 0 - 1 - (-1) = 0$

Span$\{\delta_a - \delta_b : a \neq b, a,b \geq 2\}$ is dim $n-2$.

As $\mathrm{Tr}(L) = 2(n-1) = 2n-2$. Last eigval λ satisfies

$2n - 2 - (n-2) = \lambda \Rightarrow \lambda = n$.

Eigvec ψ_n is orth to $\delta_a - \delta_b$, so $\psi_n(a) = \psi_n(b)$ $\forall a,b \geq 2$

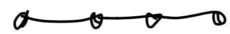
orth to $\mathbf{1}$ $\Rightarrow \psi_n(1) + (n-1)\psi_n(2) = 0$

$\Rightarrow \psi_n(1) = -(n-1)$, $\psi_n(a) = 1$ $a \geq 2$
is a solution

For $S \subset V$, ~~if~~ $1 \notin S$ $|\partial(S)| = |S|$

if $1 \in S$, $|\partial(S)| = |V-S|$

$\lambda_2 = 1$ is still good estimate of $\theta(S)$

Path P_n. $V = \{1..n\}$ $E = \{(i,i+1)\}$ 

exact later. $\theta(P_n) = \frac{2}{n}$ $S = \{1..\frac{n}{2}\}$

$|\partial(S)| = 1$

But, λ_2 very small.

Consider $x(a) = 2a-(n+1)$

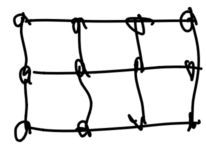
$$\sum_a x(a) = \left(2\sum_{a=1}^{n} a\right) - n(n+1) = 2\binom{n+1}{2} - n(n+1) = 0$$

For an edge $(a,a+1)$ $(x(a)-x(a+1))^2 = 2^2 = 4$

So, $x^T L x = 4(n-1)$

$$x^T x = \sum_{a=1}^{n} (2a-(n+1)) = (n+1)n\cdot(n-1)/3$$

$$\text{So, } \lambda_2 \le \frac{x^T L x}{x^T x} = \frac{4(n-1)}{(n+1)n(n-1)/3} = \frac{12}{n(n+1)}$$

<u>Products</u> <u>Ex:</u> grid 

is $P_3 \times P_4$

BD1

<u>Def</u> For $G=(V,E)$ $H=(W,F)$ $G \times H$ has vertex set $V \times W$ and edges

$$\{((a,b),(\hat{a},b)) : b \in W, \ (a,\hat{a}) \in E\}$$

$$\cup \{((a,b),(a,\hat{b})) : a \in V, \ (b,\hat{b}) \in F\}$$

<u>Thm</u> If G has eigvals $\lambda_1 \ldots \lambda_n$ with eigvecs $\alpha_1, \ldots, \alpha_n$ and H has eigvals $\mu_1 \ldots \mu_m$ with eigvecs $\beta_1 \ldots \beta_m$, then

$G \times H$ has eigval $\lambda_i + \mu_j$ for $1 \le i \le n$, $1 \le j \le m$

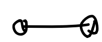
with eigvec $\gamma_{i,j}(a,b) = \alpha(a)\beta(b)$

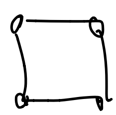
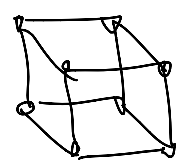
proof

$$\left(L_{G\times H}\,\gamma_{i,j}\right)(a,b) = \sum_{(a,\hat{a})\in E}\left(\gamma_{ij}(a,b)-\gamma_{ij}(\hat{a},b)\right) + \sum_{(b,\hat{b})\in F}\left(\gamma_{ij}(a,b)-\gamma_{ij}(a,\hat{b})\right)$$

$$= \sum_{(a,\hat{a})\in E}\beta_j(b)\left(\alpha_i(a)-\alpha_i(\hat{a})\right) + \sum_{(b,\hat{b})\in F}\alpha_i(a)\left(\beta_j(b)-\beta_j(\hat{b})\right)$$

$$= \beta_j(b)\cdot\alpha_i(a)\cdot\lambda_i + \alpha_i(a)\,\beta_j(b)\cdot\mu_j = \gamma_{ij}(a,b)\left(\lambda_i+\mu_j\right)$$

Ex. Hypercube. $H_1 = P_2 =$ 

$H_d = H_{d-1}\times H_1$   etc.

$L_{H_1}\begin{pmatrix}1\\1\end{pmatrix} = 0 \qquad L_{H_1}\begin{pmatrix}1\\-1\end{pmatrix} = 2\begin{pmatrix}1\\-1\end{pmatrix}$

By induction, $H_{d-1}\psi = \lambda\psi \Rightarrow H_d\begin{pmatrix}\psi\\\psi\end{pmatrix} = \lambda\begin{pmatrix}\psi\\\psi\end{pmatrix}$

and $H_d\begin{pmatrix}\psi\\-\psi\end{pmatrix} = (\lambda+2)\begin{pmatrix}\psi\\-\psi\end{pmatrix}$

Index vertices by $x \in \{0,1\}^d$

And eigvecs by $y \in \{0,1\}^d$

Now, $\psi_y(x) = (-1)^{x^T y}$, has eigval $2 \sum_i y(i)$

$$y \in \{0,1\}^{d-1} \to \begin{matrix} \psi_y \\ \psi_y \end{matrix} = \psi_{(y,0)} \qquad \begin{matrix} \psi_y \\ -\psi_y \end{matrix} = \psi_{(y,1)}$$

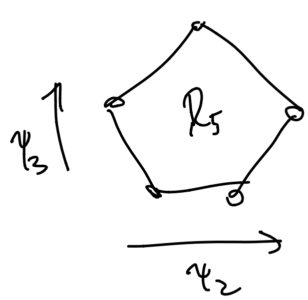
$$\lambda_2(H_d) = 2 \Rightarrow \frac{|\partial(S)|}{|S|} \geq \lambda_2 \left(1 - \frac{|S|}{n}\right)$$

For $S = \{x : x(d) = 0\}$, $|\partial(S)| = |S| = 2^{d-1} = \frac{n}{2}$

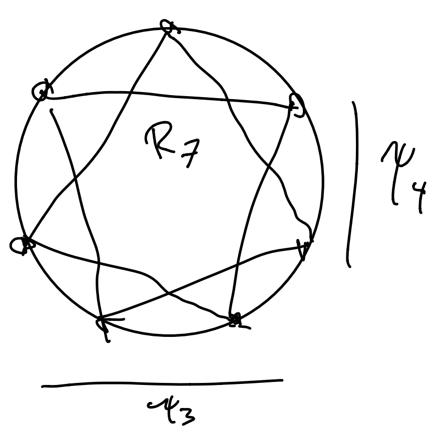
so $\lambda_2\left(1 - \frac{|S|}{n}\right) = 1$ is tight.

Ring. R_n. $V = \{0, \dots, n-1\}$ mod n

$E = \{(a, a+1) \bmod n\}$



First two eigvecs come from drawing on a regular n-gon



R_n has eigvecs

$$x_k(a) = \cos\left(\frac{2\pi k a}{n}\right)$$

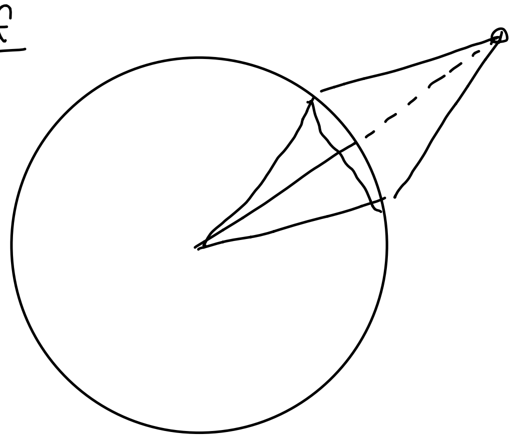
$$y_k(a) = \sin\left(\frac{2\pi k a}{n}\right)$$

of eigvals $2 - 2\cos\left(\frac{2\pi k}{n}\right)$

for $1 \le k < \frac{n}{2}$

Also, is an $x_0 = \mathbb{1}$ and $x_{\frac{n}{2}}$ if n is even.

<u>proof</u>



Or, $(L_{R_n} x_k)(a) = 2x_k(a) - x_k(a+1) - x_k(a-1)$

$$= 2\cos\left(\frac{2\pi k a}{n}\right) - \cos\left(\frac{2\pi k a}{n}\right)\cos\left(\frac{2\pi k}{n}\right) + \cancel{\sin\left(\frac{2\pi k a}{n}\right)\sin\left(\frac{2\pi k}{n}\right)}$$

$$- \cos\left(\frac{2\pi k a}{n}\right)\cos\left(\frac{2\pi k}{n}\right) - \cancel{\sin\left(\frac{2\pi k a}{n}\right)\sin\left(\frac{2\pi k}{n}\right)}$$

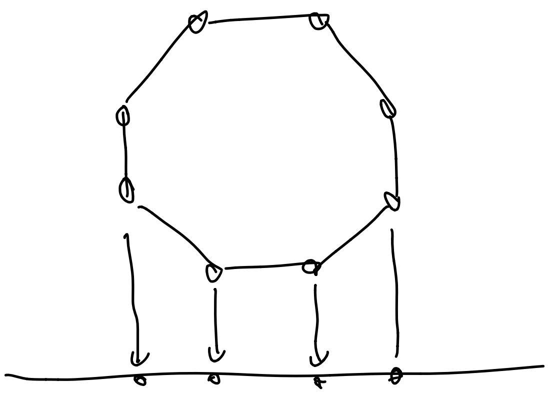
$$= \cos\left(\frac{2\pi k a}{n}\right)\left(2 - 2\cos\left(\frac{2\pi k}{n}\right)\right) = x_k(a)\cdot\left(2 - 2\cos\frac{2\pi k}{n}\right)$$

For the path, P_n has same eigvals as R_{2n},
excluding 2.

Eigvals $2\left(1-\cos\left(\frac{\pi k}{n}\right)\right)$ eigvec

$$v_k(a) = \cos\left(\frac{\pi k a}{n} - \frac{\pi k}{2n}\right)$$

Proof



If order vertices correctly, $\begin{pmatrix} I_n \\ I_n \end{pmatrix}^T L_{R_{2n}} \begin{pmatrix} I_n \\ I_n \end{pmatrix} = 2 L_{P_n}$

So, if ψ is eigvec of $L_{R_{2n}}$ s.t. $\psi(a) = \psi(a+n)$

Then $\phi = \psi(1..n)$ is eigvec of P_n of same eigval.

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A word about Cayley graphs.