Independent Sets, Coloring, and Partitioning.

(Smallest and Largest Eigenvalues)

I have to leave promptly at the end of class.
So please ask questions during lecture.

$S \subseteq V$ is <u>independent</u> if $a, b \in S \Rightarrow (a,b) \notin E$

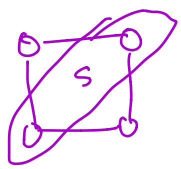
$\alpha(G) = \max\{|S| : S \text{ is independent}\}$

<u>Thm</u> $|S| \le n\left(1 - \frac{d_{ave}(S)}{\lambda_n}\right)$ where $d_{ave}(S) = \frac{d(S)}{|S|}$

ave degree in S

$\lambda_n =$ Largest Laplacian Eigval.

<u>Ex.</u>



$|S| = 2, \quad d_{ave}(S) = 2, \quad \lambda_n = 4$

$n\left(1 - \frac{d_{ave}(S)}{\lambda_n}\right) = 2$

proof

$$\lambda_n = \max_x \frac{x^T L x}{x^T x}.$$

For $S \subseteq V$, consider $1_S(a) = \begin{cases} 1 & a \in S \\ 0 & \text{o.w.} \end{cases}$

$$x = 1_S - \sigma 1 \qquad \sigma = \frac{|S|}{n}$$

So, $x(a) = \begin{cases} 1 - \sigma & a \in S \\ -\sigma & \text{o.w.} \end{cases}$

$$x^T L x = 1_S^T L 1_S = \sum_{a\sim b} \left(1_S(a) - 1_S(b)\right)^2 = d(S)$$

sum of degrees in S

$$x^T x = |S| \cdot (1-\sigma)^2 + (n-|S|)(-\sigma)^2 = n(\sigma - \sigma^2)$$

is minimum possible among vectors of form $1_S - t\mathbb{1}$

So, $$\lambda_n \geq \frac{x^T L x}{x^T x} = \frac{d(S)}{n(\sigma-\sigma^2)} = \frac{d(S)}{|S|(1-\sigma)} = \frac{d_{ave}(S)}{1-\sigma}$$

$$\Rightarrow 1-\sigma \geq \frac{d_{ave}(S)}{\lambda_n} \Rightarrow \sigma \leq 1 - \frac{d_{ave}(S)}{\lambda_n}$$

$$|S| \leq n\left(1 - \frac{d_{ave}(S)}{\lambda_n}\right)$$

Chromatic number. $\chi(G)$. Vertices of one color in a k-coloring are indep. So, $\alpha(G) \geq \frac{n}{k} \geq \frac{n}{\chi(G)}$

$$\chi(G) \geq \frac{n}{\alpha(G)}.$$

Can apply prev bound when d-regular:

$$\alpha(G) \leq n\left(1 - \frac{d}{\lambda_n}\right).$$

So, $$\chi(G) \geq \frac{1}{1 - \frac{d}{\lambda_n}} = \frac{\lambda_n}{\lambda_n - d}$$

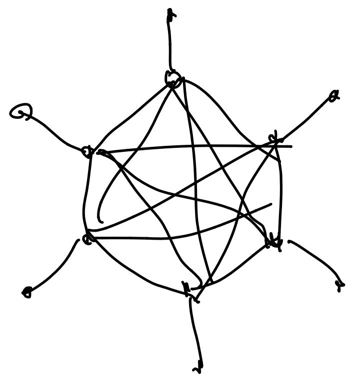
In adj mat, with $\mu_n = d - \lambda_n$, $\mu_1 = d$, is

$$\chi(G) \geq \frac{\mu_1 - \mu_n}{-\mu_n} = 1 + \frac{\mu_1}{-\mu_n}$$

Hoffman proved true even if not regular.

Thm $\chi(G) \geq 1 + \frac{\mu_1}{-\mu_n}$

Ex.



$n = 12$, $\alpha(G) = 6$

$\chi(G) = 6$

$\mu_1 = 5.19$

$\mu_n = -1.62$

$\chi(G) \geq 4.2$

adjusting edge weights -
doubling in clique, gets to 6.

Uses lem

let $M = \begin{bmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,k} \\ M_{2,1} & M_{2,2} & & \\ \vdots & & & \\ M_{k,1} & & & \end{bmatrix}$

be a symmetric matrix with blocks $M_{i,j}$

Then $(k-1)\lambda_{min}(M) + \lambda_{max}(M) \leq \sum_i \lambda_{max}(M_{i,i})$

proof of Thm

If G is k-colorable, can write

$$M = \begin{bmatrix} 0 & M_{1,2} & & \\ M_{2,1} & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$$

by making color classes contiguous.

$M_{i,i} = 0$, so $\lambda_{max}(M_{i,i}) = 0$

$\Rightarrow$ $(k-1)\lambda_{min}(M) + \lambda_{max}(M) \leq 0$

$(k-1)\mu_n + \mu_1 \leq 0$

$k\mu_n \leq \mu_n - \mu_1$

$k \geq \frac{\mu_n}{\mu_n} + \frac{\mu_1}{-\mu_n} = 1 + \frac{\mu_1}{-\mu_n}$, as $\mu_n < 0$

Note: last class proved Lem for $k=2$ and 0 diagonal.
More general case uses induction and interlacing.

Cuts, Clusters, and Partitions

Boundary of $S \subseteq V$, $\partial(S) = \{(a,b) \in E : a \in S, b \notin S\}$

isoperimetric ratio $\theta(S) = \dfrac{|\partial(S)|}{|S|}$

want S with $\theta(S)$ small, $|S|$ not bigger than half

$$\theta_G = \min_{|S| \leq \frac{n}{2}} \theta(S)$$

Thm For all $S \subset V$ $\quad \theta(S) \geq \lambda_2(1-\sigma) \qquad \sigma = \frac{|S|}{n}$

In particular $\theta_G \geq \lambda_2/2$.

proof $\quad \lambda_2 = \min\limits_{x^T 1 = 0} \dfrac{x^T L x}{x^T x}$

Consider $x = 1_S - \sigma 1$. $\quad x^T 1 = 1_S^T 1 - \sigma 1^T 1$

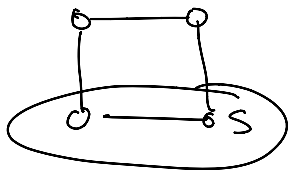
$$= |S| - \sigma n = 0$$

So, $\lambda_2 \leq \dfrac{x^T L x}{x^T x} \qquad x^T x = n(\sigma - \sigma^2) = |S|(1-\sigma)$

$$x^T L x = \sum_{(a,b) \in E} (x(a) - x(b))^2 = |\partial(S)|$$

$$\lambda_2 \leq \frac{|\partial(S)|}{|S|(1-\sigma)} \Rightarrow \theta(S) \geq \lambda_2 (1-\sigma)$$

Example



$\lambda_2 = 2$

$\partial(S) = 2$ $\quad$ $\sigma = \frac{1}{2}$

$|S| = 2$ $\quad$ $\lambda_2(1-\sigma) = 1$

$\theta(S) = 1$ $\quad$ tight

Cheeger's inequality will show almost tight.

Mention Local Clustering.

Conductance: count vertices by degree

$$d(S) = \sum_{a \in S} d(a).$$

$$\phi(S) = \frac{|\partial(S)|}{\min(d(S), d(V-S))}$$

$$\phi_G = \min_{S \subset V} \phi(S)$$

Want to relate to generalized Rayleigh quotient

$$\frac{y^T L y}{y^T D y}$$

If set $y = D^{-1/2} x$,

$$= \frac{x^T D^{-1/2} L D^{-1/2} x}{x^T x}$$

Normalized Laplacian

$$N \stackrel{\text{def}}{=} D^{-1/2} L D^{-1/2} = D^{-1/2}(D - M) D^{-1/2} = I - D^{-1/2} M D^{-1/2}$$

Set eigvals to be $0 = \nu_1 \le \nu_2 \le \dots \le \nu_n$

Eigvec of 0 is $d^{1/2}$: $d^{1/2}(a) = \sqrt{d(a)}$

$$D^{-1/2} L D^{-1/2} d^{1/2} = D^{-1/2} L \mathbf{1} = D^{-1/2} 0 = 0.$$

$$\nu_2 = \min_{x : x^T d^{1/2} = 0} \frac{x^T N x}{x^T x}$$

Translating back to y gives $\quad x = D^{1/2} y$

$$x^T d^{1/2} = y^T D^{1/2} d^{1/2} = y^T d$$

$$\nu_2 = \min_{y : y^T d = 0} \frac{y^T L y}{y^T D y}$$

Theorem $\forall S$, $\dfrac{|\partial(S)| \cdot d(V)}{d(S) \cdot d(V-S)} \ge \nu_2 \implies \phi(S) \ge \nu_2 / 2$

Similar proof