Notation: $a \sim b$ means $(a,b) \in E$

For $S \subseteq V$, $G(S)$ is the induced subgraph on S.

It has vertex set S and edges $\{(a,b) \in E : a \in S \text{ and } b \in S\}$

Adjacency matrices, eigenvalue interlacing, PF theory.

$$M(a,b) = \begin{cases} w_{a,b} & \text{if } (a,b) \in E \\ 0 & \text{o.w.} \end{cases} \qquad w_{a,b} > 0$$

Denote eigvals $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$

Reason: for d-regular, $L = dI - M$

so $\lambda_i = d - \mu_i$ $\qquad \lambda_1 = 0 \leq \lambda_2 \leq \dots \leq \lambda_n$

For irregular, μ_1 and corresp eigvec is more interesting.

<u>Thm 1</u> $\quad d_{ave} \leq \mu_1 \leq d_{max}$.

<u>proof</u> By CF

$$\mu_1 = \max_x \frac{x^T M x}{x^T x} \geq \frac{\mathbf{1}^T M \mathbf{1}}{\mathbf{1}^T \mathbf{1}} = \frac{\mathbf{1}^T d}{n} = \frac{\sum_{a \in V} d(a)}{n} = d_{ave}$$

Let ψ_1 be eigvec of μ_1

Let $a = \arg\max \psi_1(a)$. Then

$$\underline{\mu_1} \cdot \psi(a) = \sum_{b \sim a} w_{a,b} \psi(b) \leq \sum_{b \sim a} w_{a,b} \psi(a) \leq d(a)\, \psi(a) \leq \underline{d_{max}}\, \psi(a)$$

μ_1 also says something about subgraphs.

Cauchy's Interlacing Theorem

Let A be a matrix and let B be a principal submatrix (obtained by deleting a row and column).

$$\text{eigs}(A) = \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n, \quad \text{eigs}(B) = \beta_1 \geq \beta_2 \geq \cdots \geq \beta_{n-1}$$

Then $\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \cdots \geq \alpha_{n-1} \geq \beta_{n-1} \geq \alpha_n$.

<u>proof:</u> Assume remove first row/col to get B.

First, $\alpha_k \geq \beta_k$.

$$CF: \alpha_k = \max_{\substack{S \dim k \\ S \subseteq \mathbb{R}^n}} \min_{x \in S} \frac{x^T A x}{x^T x}$$

$$\beta_k = \max_{\substack{S \dim k \\ S \subseteq \mathbb{R}^{n-1}}} \min_{x \in S} \frac{x^T B x}{x^T x} = \max_{\substack{S \subseteq \mathbb{R}^{n-1} \\ S \dim k}} \min_{x \in S} \frac{\begin{pmatrix} 0 \\ x \end{pmatrix}^T B \begin{pmatrix} 0 \\ x \end{pmatrix}}{\begin{pmatrix} 0 \\ x \end{pmatrix}^T \begin{pmatrix} 0 \\ x \end{pmatrix}}$$

$$\leq \alpha_k$$

$\alpha_{k+1} \leq \beta_k$ follows by changing A and B to $-A$ and $-B$

If remove a vertex, obtain submatrix as adj matrix.
μ_1 can only go down, while dave can go up or down.

<u>Prop 2</u> For every $S \subseteq V$, $d_{ave}(G(S)) \leq \mu_1$.
<u>proof</u> Cauchy $\Rightarrow$ max eig of $M(S) \leq \mu_1$,
and can apply Thm 1

easier proof by $\frac{1^T_S M 1_S}{1^T_S 1^T_S}$ ...

Profit: Graph coloring.

A k-coloring of G is $c: \{1..k\} \to V$ s.t.

$\forall (a,b) \in E$, $c(a) \neq c(b)$.

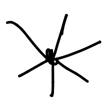
$\chi(G) = \min \{k : G \text{ is } k\text{-colorable}\}$.

Easy: $\chi(G) \le d_{max} + 1$, by greedy.

Wilf: $\chi(G) \le \lfloor \mu_1 \rfloor + 1$.

note μ_1 can be much less than d_{max}.

 $\mu_1 = \sqrt{2}$, $d_{max} = 2$

 with $n+1$ vertices, $\mu_1 = \sqrt{n}$, $d_{max} = n$

Proof. by induction on $|V|$. Base case $|V| = 1$.

$\chi(G) = 1$, $\mu_1 = 0$.

Induction: we know $d_{min} \le d_{ave} \le \mu_1$. So

G has a vertex, a, with $d(a) \le \lfloor \mu_1 \rfloor$. Let $S = V - \{a\}$.

Cauchy $\Rightarrow$ $\mu_1(G(S)) \le \mu_1(G)$, so induct hyp $\Rightarrow$

can color $G(S)$ with $\lfloor \mu_1 \rfloor + 1$ colors.

Remains to pick a color for a. It has $\le \lfloor \mu_1 \rfloor$ neighbors,

so a color is available.

PF Theory The eigenvector of μ_1.

Thm let G be a connected weighted graph.

a. μ_1 has a strictly positive eigenvector.

b. $\mu_1 \geq |\mu_n|$

c. $\mu_1 > \mu_2$

Lem 3 let G be connected and let ψ be a non-negative eigenvector of M. Then ψ is strictly positive.

proof Assume, twoc, $\exists$ a s.t. $\psi(a)=0$.

As G connected $\Rightarrow \exists (a,b)$ s.t. $\psi(a)=0<\psi(b)$.

let μ be the eigval. Then

$$\mu \cdot \psi(a) = \sum_{z\sim a} M(a,z)\,\psi(z) \geq M(a,b)\,\psi(b) > 0, \;\#$$

proof of a.

let $\psi_1 = $ eigvec of $\overset{\text{norm } 1}{\mu_1}$, $x(a) = |\psi_1(a)|\ \forall a$.

Will show x is eigvec of μ_1. Lem 3 $\Rightarrow$ strictly positive

$$\mu_1 = \psi_1^T M \psi_1 = \sum_{(a,b)\in E} M(a,b)\,\psi_1(a)\,\psi_1(b)$$

$$\leq \sum_{a\sim b} M(a,b)\,|\psi_1(a)|\cdot|\psi_1(b)| = x^T M x \leq \mu_1$$

Now, use maximum $\Rightarrow$ eigenvector.

proof of b. Let ψ_n = eigvec of μ_n, $y(a) = |\psi_n(a)|$, $\forall a$.

$$|\mu_n| = |\psi_n^T M \psi_n| \le \sum_{a \sim b} M(a,b)\, y(a) y(b) \le y^T M y \le \mu_1 \qquad (*)$$

keep

proof of c. $\mu_2 < \mu_1$.

Let ψ_2 be eigvec of μ_2. As $\psi_2 \perp \psi_1$, ψ_2 has pos and neg entries. Let $y(a) = |\psi_2(a)|$, $\forall a$.

$$\mu_2 = \psi_2^T M \psi_2 \le y^T M y \le \mu_1$$

If $\mu_2 = \mu_1$, y is nonneg eigvec of μ_1, and thus strictly positive $\Rightarrow$ ψ_2 is never zero

$\Rightarrow \exists (a,b)$ s.t. $\psi_2(a) < 0 < \psi_2(b)$.

$\Rightarrow \psi_2^T M \psi_2 < y^T M y$, as $M(a,b)\psi_2(a)\psi_2(b) < 0$

$< M(a,b)\, y(a)\, y(b)$.

Contradiction.

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what if $\mu_n = -\mu_1$?

Thm If G connected, $\mu_n = -\mu_1$ iff G is bipartite.

If $\mu_n = -\mu_1$, $*$ is tight

$\Rightarrow$ y an eigvec of μ_1, strictly positive,

$\Rightarrow$ ψ_n is never zero, and $\exists (a,b)$ s.t. $\psi_n(a) < 0 < \psi_n(b)$

$$\Rightarrow \left| \underbrace{\sum_{a\sim b} M(a,b)\, \psi_n(a)\, \psi_n(b)} \right| = \sum_{a\sim b} M(a,b)\, |\psi_n(a)| \cdot |\psi_n(b)|$$

$\Rightarrow$ all term in have same sign. Edge $(a,b) \Rightarrow$ negative.

So, $\psi_n(a)\,\psi_n(b) < 0$ all $(a,b) \in E$

$\Rightarrow$ signs give bipartition.

Bipartite $\Rightarrow$ μ eigval $\Longleftrightarrow$ $-\mu$ eigval.

can order vertices so $M = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$

let $M\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \mu \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$, so $\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} Bx_1 \\ B^T x_0 \end{pmatrix} = \begin{pmatrix} \mu x_0 \\ \mu x_1 \end{pmatrix}$

Then $M\begin{pmatrix} x_0 \\ -x_1 \end{pmatrix} = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ -x_1 \end{pmatrix} = \begin{pmatrix} -Bx_1 \\ Bx_0 \end{pmatrix} = \mu\begin{pmatrix} -x_0 \\ x_1 \end{pmatrix} = -\mu\begin{pmatrix} x_0 \\ -x_1 \end{pmatrix}$

So, $-\mu$ is an eigenvalue.