H is an ε-approx of G if

$$(1-\varepsilon) G \preceq H \preceq (1+\varepsilon) G$$

weighted expander is ε-approx of complete graph

<u>Today</u> every weighted graph G has an ε-approx H with $\leq 4 n \ln n$ edges

---

Implications. As $1_S^T L_G 1_S = w(\partial(S))$

boundary of every set has similar weight.

$1_{\{a\}}^T L_G 1_{\{a\}} = d(a)$.

So weighted degrees are similar.

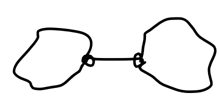
$\Rightarrow$ if have fewer edges they must have higher weight

<u>Thm</u> (next week) $L_H \preceq L_G \iff L_G^{+} \preceq L_H^{+}$

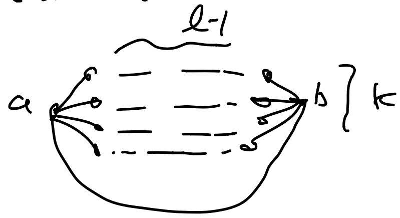
so inverses similar

Will have edges of H $\subseteq$ edges of G.

$\Rightarrow$ cut edges must be in H



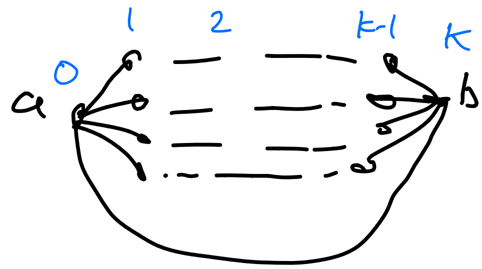
<u>Ex.</u>



k paths length $l+1$

If $l > k$, must include edge (a,b)

Consider function



sum is $\underbrace{k^2}_{a,b} + \underbrace{(k+1)\, l \cdot 1^2}_{\text{paths}}$

if $k > l$, most of energy is on this edge.

Random sampling: set $p_{a,b}$ for each edge a,b

include with prob $p_{a,b}$,

if do so, make weight $\frac{w_{a,b}}{p_{a,b}}$

$$L_G = \sum_{(a,b) \in E} w_{a,b} L_{a,b}$$

$$\mathbb{E}\, L_H = \sum_{(a,b) \in E} p_{a,b} \frac{w_{a,b}}{p_{a,b}} L_{a,b} = L_G$$

Need to bound deviations

Standard: Chernoff Bounds

For random variables $x_1, \dots, x_n \in [0, R]$ $\quad \mu = \mathbb{E} \sum x_i$

$$\Pr\left[\sum x_i > (1+\varepsilon)\mu \right] \le e^{-\frac{\varepsilon^2 \mu}{3R}}$$

$$\Pr\left[\sum x_i < (1-\varepsilon)\mu \right] \le e^{-\frac{\varepsilon^2 \mu}{2R}}$$

Matrix Chernoff

For random PSD matrices $X_1, \dots, X_m$, $\|X_i\| \le R$

$$M = \mathbb{E} \sum X_i$$

$$\Pr\left[\mu_{max}\left(\sum X_i \right) \ge (1+\varepsilon) \mu_{max}(M) \right] \le n e^{-\varepsilon^2/3R}$$

$$\Pr\left[\mu_{min}\left(\sum X_i \right) \le (1-\varepsilon) \mu_{min}(M) \right] \le n e^{-\varepsilon^2/2R}$$

Transform

For invertible C, $\quad A \preccurlyeq B \iff CAC^T \preccurlyeq CBC^T$

So, $\quad L_H \preccurlyeq (1+\varepsilon) L_G \iff L_G^{+/2} L_H L_G^{+/2} \preccurlyeq L_G^{+/2} L_G L_G^{+/2}$

$$= \Pi = \tfrac{1}{n} L_{K_n}$$

Π is projection, identity orthogonal to $\mathbb{1}$.

$$\mathbb{E} L_H = L_G \implies \mathbb{E}\, L_G^{+/2} L_H L_G^{+/2} = \Pi$$

Apply Matrix Chernoff to this, ignoring span$(\mathbb{1})$

So, $\mu_{max}(M) = 1$, and orth $\mathbb{1}$, $\mu_2 \ (M) = 1$

$$X_{a,b} = \begin{cases} \frac{w_{a,b}}{p_{a,b}} L_G^{+/2} L_{a,b} L_G^{+/2} & \text{w/ prob } p_{a,b} \\ 0 & \text{o.w.} \end{cases}$$

Choose $p_{a,b} = \frac{1}{R} w_{a,b} \| L_G^{+/2} L_{a,b} L_G^{+/2} \|$ R tbd

so, $\| X_{a,b} \| \leq R$

Best bound when make this uniform

Try $R = \frac{\varepsilon^2}{3.5 \ln n}$

Then error prob is $\leq 2n e^{-\frac{\varepsilon^2}{3R}} = 2n e^{-\frac{3.5}{3} \ln n}$

$= 2 n^{-1/6} \to 0$

$\mathbb{E} \# \text{edges} = \sum_{(a,b)} p_{a,b}$

$$\| L_G^{+/2} L_{a,b} L_G^{+/2} \| = \| L_G^{+/2} (\delta_a - \delta_b)(\delta_a - \delta_b)^T L_G^{+/2} \|$$

$$= (\delta_a - \delta_b)^T L_G^{+} (\delta_a - \delta_b) = \mathrm{Reff}(a,b)$$

$$p_{a,b} = \frac{1}{R} \cdot w_{a,b} \cdot \mathrm{Reff}(a,b)$$

Thm $\sum_{(a,b)\in E} w_{a,b} \, R_{\mathrm{eff}}(a,b) = n-1$

$$= \sum_{a\sim b} w_{a,b} (\delta_a - \delta_b)^T L_G^+ (\delta_a - \delta_b)$$

$$= \sum_{a\sim b} w_{a,b} \operatorname{Tr}\left[(\delta_a - \delta_b)^T L_G^+ (\delta_a - \delta_b) \right]$$

$$= \sum_{a\sim b} w_{a,b} \operatorname{Tr}\left[(\delta_a - \delta_b)(\delta_a - \delta_b)^T L_G^+ \right]$$

$$= \operatorname{Tr}\left[\left(\sum_{a\sim b} w_{a,b} (\delta_a - \delta_b)(\delta_a - \delta_b)^T \right) L_G^+ \right]$$

$$= \operatorname{Tr}\left[L_G L_G^+ \right] = n-1$$

$$\Rightarrow \mathbb{E} \# \text{ edges} = \frac{n-1}{R} \leq \frac{3.5 \, n \ln n}{\varepsilon^2}$$

Chernoff says unlikely to be more than $\frac{4 n \ln n}{\varepsilon^2}$ edges.

What if $p_{a,b} > 1$?

Set $p_{a,b} = \min\left(1, \frac{1}{R} w_{a,b} \, R_{eff}(a,b)\right)$

include edge (a,b) with prob 1,
redo analysis on rest of graph without this edge.

or, subdivide the edge into $\lceil p_{a,b} \rceil$ parallel edges by dividing weight equally.

Now, all $p_{a,b} \le 1$, and little increase in $\mathbb{E}$ # edges.

Note can quickly estimate R_{eff} !

New technique: short cycle decomposition.