Expanders: Goal: for all $\varepsilon > 0$ find d and an infinite sequence of d-regular graphs that are ε-expanders

Even better, given a vertex v, want to enumerate its neighbors in time polynomial in $\log n$.

Will do it recursively.

Need to assume some expander exists.

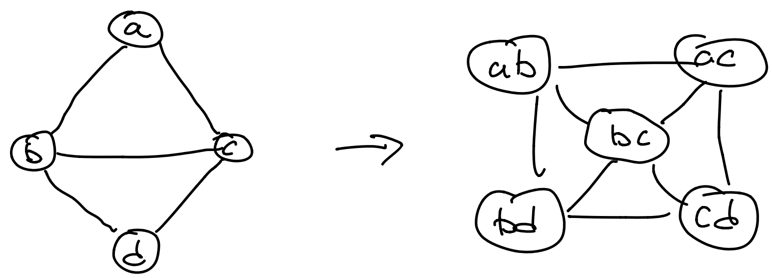
Thm $\exists f(\varepsilon)$ s.t. $\forall \varepsilon > 0$, for all suff large n, $\exists$ an $f(\varepsilon)$-regular ε-expander on n vertices

easy if say $O(f(\varepsilon) \log n)$

	n	d	ε
line-graph	$n \to \frac{dn}{2}$	$d \to 2(d-1)$	—
clique → expander	—	$\Downarrow$	$\uparrow$ a little
squaring	—	$d \to d(d-1)$	$\Downarrow$

line graph of $G = (V, E)$ is $H(E, F)$

$F = \{ ((a,b), (b,c)) \text{ s.t. } a \neq c \}$



If G is d-regular, H is $2(d-1)$ regular.

every vertex of H is contained in two d-cliques, one for each vertex.

Algebraic def: recall U = signed edge-vertex adj mat

$$U((a,b), c) = \begin{cases} 1 & a = c \\ -1 & b = c \\ 0 & \text{o.w.} \end{cases}$$

$L = U^T U$. $\quad M_H = |U| \, |U|^T - 2 I_E$

$|U|^T |U| = dI + M_G$

<u>Thm</u> eigs of L_H = eigs of $L_G \cup \{0\}^{\frac{dn}{2} - n}$

$$\mathrm{eigs}\left(|U|^T|U|\right) = \mathrm{eigs}\left(|U|\cdot|U|^T\right) \cup \{0\}^{\frac{dn}{2}-n}$$

proof λ_i eigval of $L_G = D_G - M_G$

$\Rightarrow d-\lambda_i$ eigval of M_G

$\Rightarrow 2d-\lambda_i$ eigval of $D_G + M_G = |U|^T|U|$

$\Rightarrow 2d-\lambda_i$ eigval of $|U||U|^T = 2I_E + M_H$

$\Rightarrow 2(d-1)-\lambda_i$ eigval of M_H

$\Rightarrow \lambda_i$ eigval of L_H

eigvals of 0 of $2I_E + M_H$

become eigvals of $2d$ in L_H

Problem: degree $d \to 2(d-1)$

Clique replacement: let Z be an α-expander on d vertices of degree k

So $(1-\alpha)\frac{k}{d}L_{K_d} \preccurlyeq Z \preccurlyeq (1+\alpha)\frac{k}{d}L_{K_d}$

H is comprised of d-cliques.

Replace each with Z, and call result $G \mathbin{Ⓛ} Z$

$G \, Ⓛ \, Z$ means: replace each d-clique in line graph of G with Z.

Thm $\quad (1-\alpha)\frac{k}{d} H \preccurlyeq G \, Ⓛ \, Z \preccurlyeq (1+\alpha)\frac{k}{d} H$

proof

let $H_1, \dots H_n$ be d-cliques in H,

$$\text{so} \quad L_H = \sum_i L_{H_i}$$

let $Z_1 \dots Z_n$ be copies of Z on same ~~vertex~~ set as H_i

$$\text{So} \quad G \, Ⓛ \, Z = \sum_i Z_i$$

$$Z_i \preccurlyeq (1+\alpha)\frac{k}{d} L_{H_i}$$

$$G \, Ⓛ \, Z = \sum_i L_{Z_i} \preccurlyeq (1+\alpha)\frac{k}{d} \sum_i L_{H_i}$$

Similar ε, same # vertices, fewer edges

Squaring: G^2 = graph s.t. $M_{G^2} = M_G^2 - dI$

(a,b) is an edge if $\exists\, c$ s.t. (a,c) and (c,b) in E.

Remove self loops

Keep multiedges

Adj mat eigvals are $(\mu_i(G))^2 - d$

degree $d(d-1)$.

How to combine it all.

Relative spectral gap:

$$\tau(G) = \min\left(\frac{\lambda_2}{d}, \frac{2d-\lambda_n}{d}\right) \qquad \text{want big}$$

focus on this

Lem $\quad \tau(G^2) \geq 2\tau(G) - \tau(G)^2$

proof

$$\lambda_2(G^2) = d(d-1) - \mu_2(G^2)$$

$$= d(d-1) - \left(\mu_2(G)^2 - d\right)$$

$$= d(d-2) - \mu_2(G)^2$$

$$\mu_2(G) = d - \lambda_2(G) \leq d(1-\tau(G))$$

$$= d(d-2) - d^2(1-\tau(G))^2$$

$$\tau(G^2) = \frac{d(d-2)}{d(d-1)} - \frac{d^2}{d(d-1)}(1-\tau(G))^2$$

$$= \frac{2d^2}{d(d-1)}\tau(G) - \frac{d^2}{d(d-1)}\tau(G)^2 + \frac{2}{d(d-1)}$$

$$\geq 2\tau(G) - \tau(G)^2$$

$$\tau(H) \leq \frac{\tau(G)}{2} \implies \tau(G \textcircled{z} Z) \geq \frac{(1-\alpha)\tau(G)}{2}$$

If $\tau(G)$ smallish,

$$\tau(G \textcircled{z} Z) \geq \frac{(1-\alpha)\tau(G)}{2}$$ also smallish

$$\tau((G \textcircled{z} Z)^2) \geq (1-\alpha)\tau(G) - \text{a little}$$

$$\tau(((G \textcircled{z} Z)^2)^2) \geq 2(1-\alpha)\tau(G) - \text{a little}$$

$$\geq \tau(G).$$

\# vertices of $((G \textcircled{z} Z)^2)^2 = \frac{dn}{2}$

want degree $= d$,

degree of $(G \textcircled{z} Z) \approx 2k$

so need $d \approx 16k^4$