Spectral and Algebraic Graph Theory 462/562

formerly grad course numbered AMTh 561/CPSC 662

Spectral = eigvals and eigvecs

Ugrad/grad => I assume less.

Grads get extra homework problems

More applied/CS focus

But still very mathy

Can find lecture notes from previous years.

Writing a book now. The book will have details I cannot cover in class.

I will distribute my handwritten notes, when I have them.

Assignments: 5-6. Probably 6. Can work in small groups (for now). No tests or exam.

Occasional recommended exercises, especially for this lecture.

Preqs: linear algebra. graph theory. some probability. proof-based exposition. endurance

Today: Intro, overview, a proof or two.

Get used to my notation.

Please interrupt when necessary.

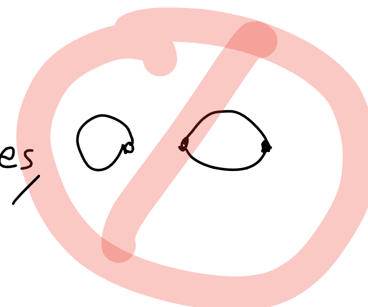
Graphs. $G = (V, E)$ E is set of pairs of V.

Write edges as (a,b), although $\{a,b\}$ would be better.

$(a,b) = (b,a)$

Undirected.

No self-loops or multi-edges usually.



Might be weights on edges. If so, almost always positive.

Sources: social networks, comm. networks, etc.

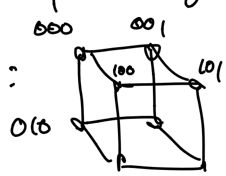
abstract like

path on n vertices. $V = \{1 \dots n\}$

$E = \{(a, a+1) : 1 \le a < n\}$

ring: path plus edge $(1,n)$

hypercube:



$V = \{0,1\}^d$

$(a,b) \in E$ if $|\{i : a(i) \neq b(i)\}| = 1$

random.

edge (a,b) appears with probability p, independently chosen.

Matrices for graphs.

Adjacency. M. row/cols labeled by V.

$$M(a,b) = \begin{cases} 1 & (a,b) \in E \\ 0 & \text{o.w.} \end{cases}$$

Is using matrix as a spreadsheet

Very surprising eigenvalues or eigenvectors should matter.

Diffusion Operator / walk Matrix.

Let D = diagonal matrix of degrees.

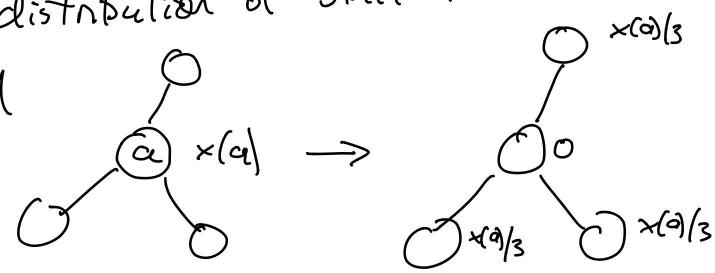
$$d = M \cdot \mathbb{1} \qquad D = \mathrm{diag}(d)$$

$$W = D^{-1} M$$

Let $p \in \mathbb{R}^V$ be a vector s.t. $p(a)$ = amount of stuff at vertex a.

If stuff at a moves to neighbors of a, evenly, then new distribution of stuff is

$$q^T = p^T D^{-1} M$$



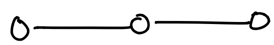
Total amount of stuff, $p^T \mathbb{1}$, is conserved because $q^T \mathbb{1} = p^T D^{-1} M \mathbb{1} = p^T D^{-1} d = p^T \mathbb{1}$

Expect spectra of W to matter.

Laplacian $\quad L = D - M$

defines a natural quadratic form:

for $x \in \mathbb{R}^V$, $\quad x^T L x = \sum_{(a,b)\in E} w_{a,b} \big(x(a) - x(b)\big)^2$

e.g. 

x: 0 $\quad$ 1 $\quad$ 3 $\qquad x^T L x = 1^2 + 2^2 = 5$

Spectral Theory

ψ is an eigvec of M of eigval λ if

$$M\psi = \lambda \psi$$

Theorem Every real symmetric n-by-n matrix M has n real eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ and n orthonormal eigenvectors $\psi_1, \ldots, \psi_n$

$$\left(\psi_i^T \psi_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{o.w.} \end{cases} \right)$$

s.t. $M\psi_i = \mu_i \psi_i$

Note: $\psi_1, \ldots, \psi_n$ not uniquely defined.

Examples in Jupyter

Course topics

Graph structure: cuts, coloring, indep sets,
partitioning, local partitioning

The zoo: fundamental examples

Estimating eigenvalues

Random walks.

Physical models: springs & resistors

Effective resistance and elimination

Expanders - extremal combinatorics
relation to codes and pseudo randomness

Sparsification

Solving Laplacian equations
and computing eigenvectors.

Def The Rayleigh Quotient of x with respect to M is $\frac{x^T M x}{x^T x}$

Theorem If M is symmetric and x maximizes $\frac{x^T M x}{x^T x}$ then $Mx = \mu_1 x$.

Will expand $x = \sum_{i=1}^{n} c_i \psi_i$, where $c_i = \psi_i^T x$

Why? well $x = \sum x(i) \delta_i$, where δ_i is elem vec in direction i, and $x(i) = \delta_i^T x$.

And, $\sum_i c_i \psi_i = \sum_i \psi_i \cdot \psi_i^T x = \left(\sum_i \psi_i \psi_i^T\right) x$

$= I x = x$

Claim $x^T M x = \sum_i c_i^2 \mu_i$, where $c_i = \psi_i^T x$

proof

$$x^T M x = \left(\sum_i c_i \psi_i\right)^T M \left(\sum_j c_j \psi_j\right)$$

$$= \left(\sum_i c_i \psi_i\right)^T \left(\sum_j c_j \lambda_j \psi_j\right)$$

as $\quad \varphi_i^T \varphi_j = \begin{cases} 1 & i=j \\ 0 & \text{o.w.} \end{cases}$

$$= \sum_i c_i^2 \lambda_i$$

<u>proof of theorem</u> for all x

$$\frac{x^T M x}{x^T x} = \frac{\sum_i c_i^2 \mu_i}{\sum c_i^2} \leq \frac{\sum_i c_i^2 \mu_1}{\sum_i c_i^2}$$

as $\quad c_i^2 \geq 0$ and $\mu_1 \geq \mu_i, \forall i$

$$= \mu_1$$

and, equality only holds if $\quad c_i^2 = 0$ for $\mu_i < \mu_1$