A d-regular graph G is an ε-expander

if $\mu_2 \le \varepsilon d$ and $\mu_n \ge -\varepsilon d$

Equiv to $|\lambda_i - d| \le \varepsilon d$ for $i \ge 2$, Laplacian

And $\|L_G - \frac{d}{n} K_n\| \le \varepsilon d$

where $\|A\| = \max_x \frac{\|Ax\|}{\|x\|}$

for sym A, $\|A\| = \max |\text{eigval}(A)|$

recall for $i \ge 2$, $\lambda_i(K_n) = n$, so $\lambda_i(\frac{d}{n} K_n) = d$

How small ε?

Ramanujan: $\varepsilon \le \frac{2\sqrt{d-1}}{d}$

Today: $\varepsilon \le \frac{2\sqrt{d-1}}{d} - c$, $c > 0$, impossible for large n, d fixed.

For $A, B \subseteq V$, $A \cap B = \emptyset$

$E(A,B) \triangleq \{(a,b) \in E, a \in A, b \in B\}$ disjoint

If G is d-regular, pick A and B at random, sizes αn, βn

$\Pr[a \in A, b \in B] = \alpha\beta$

$\Pr[(a,b) \in E(A,B)] = 2\alpha\beta$

$\mathbb{E}|E(A,B)| = \frac{dn}{2} \cdot 2\alpha\beta = d\alpha\beta n$

For K_n, $|A| = \alpha n$, $|B| = \beta n$

$$E(A,B) = (\alpha n)(\beta n) = \alpha\beta n^2, \quad \text{but } d = n-1$$

<u>Thm</u> $\forall A, B$ s.t. $A \cap B = \phi$,

$$|E(A,B) - d\alpha\beta n| \leq \varepsilon d n \sqrt{\alpha - \alpha^2}\sqrt{\beta - \beta^2}$$

is small error when $\alpha\beta > \varepsilon$

<u>proof</u> $1_A^T L_G 1_B = -E(A,B)$

let $H = \frac{d}{n} K_n$ $\quad 1_A^T L_H 1_B = -\frac{d}{n}|A| \cdot |B| = d\alpha\beta n$

$$|1_A^T (L_G - L_H) 1_B| = |E(A,B) - d\alpha\beta n|$$

$\leq \|1_A\| \cdot \|(L_G - L_H) 1_B\|$ is Cauchy-Schwartz

$\leq \|1_A\| \cdot \|L_G - L_H\| \cdot \|1_B\|$ by norm

$\leq \sqrt{\alpha n}\sqrt{\beta n}\, \varepsilon d = d\sqrt{\alpha}\sqrt{\beta}\, \varepsilon n$

to improve, let $x_A = 1_A - \alpha \mathbb{1}$ $\quad x_B = 1_B - \beta \mathbb{1}$,

now, $1_A^T (L_G - L_H) 1_B = x_A^T (L_G - L_H) x_B$,

$\|x_A\| = \sqrt{n(\alpha - \alpha^2)}$ $\quad \|x_B\| = \sqrt{n(\beta - \beta^2)}$

gives the result.

For $A \subset V$, $N(A) = \{b : \exists a \in A, (a,b) \in E\}$

<u>Tanner's Theorem</u>

For $|A| = \alpha n$, $|N(A)| = \gamma n$, $\gamma \geq \dfrac{\alpha}{\varepsilon^2(1-\alpha) + \alpha}$

For α small, think of as $\dfrac{\alpha}{\varepsilon^2}$ for $\varepsilon \approx \dfrac{2\sqrt{d-1}}{d}$

is like $\alpha \dfrac{d^2}{4(d-1)} \approx \alpha \dfrac{d}{4}$

<u>proof:</u> $T = V - N(A)$ so no edges between T and A

$\beta n = |T|$. $\beta = 1 - \gamma$

$$\alpha \beta d n \leq \varepsilon d n \sqrt{\alpha(1-\alpha)\beta(1-\beta)}$$

$$\sqrt{\alpha\beta} \leq \varepsilon \sqrt{(1-\alpha)(1-\beta)}$$

$$\alpha\beta \leq \varepsilon^2 (1-\alpha)(1-\beta)$$

$$\frac{\beta}{1-\beta} \leq \frac{\varepsilon^2(1-\alpha)}{\alpha}$$

$$\frac{1-\gamma}{\gamma} \leq \frac{\varepsilon^2(1-\alpha)}{\alpha}$$

$$\frac{1}{\gamma} \leq \frac{\varepsilon^2(1-\alpha)}{\alpha} + 1$$

$$\gamma \geq \frac{\alpha}{\varepsilon^2(1-\alpha) + \alpha}$$

How small can ε be?

For $|A|=1$, $\alpha = \frac{1}{n}$, $|N(A)| = d$, $\gamma = \frac{d}{n}$, $\varepsilon \geq 1/\sqrt{d}$

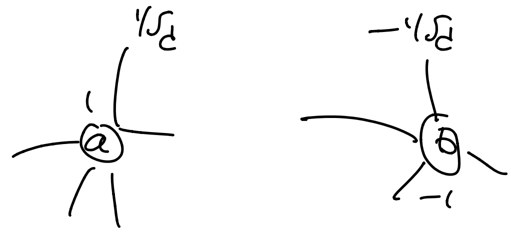
Will show for large graphs is an $x \perp \mathbb{1}$ s.t.

$$\frac{x^T L x}{x^T x} \leq d - 2\sqrt{d-1} + \text{small}$$

Let $a, b \in V$ $\quad N(a) \cap N(b) = \emptyset$

and no edges between $N(a)$ and $N(b)$

$$x(c) = \begin{cases} 1 & \text{if } c = a \\ -1 & c = b \\ 1/\sqrt{d} & c \in N(a) \\ -1/\sqrt{d} & c \in N(b) \end{cases}$$



$x \perp \mathbb{1}$, so $\quad \lambda_2 \leq \dfrac{x^T L x}{x^T x}$

$$= \frac{d\left(1 - \frac{1}{\sqrt{d}}\right)^2 + d(d-1)\left(\frac{1}{\sqrt{d}}\right)^2 + \text{same}}{1^2 + d\left(\frac{1}{\sqrt{d}}\right)^2 + \text{same}}$$

$$= \frac{d - 2\sqrt{d} + 1 + d - 1}{2} = d - \sqrt{d}$$

We already knew that!

Improve by basing ans on far apart edges

Assume are edges (a_0, a_1), (b_0, b_1) at distance $\geq 2k+2$

If d fixed, n big, this happens.

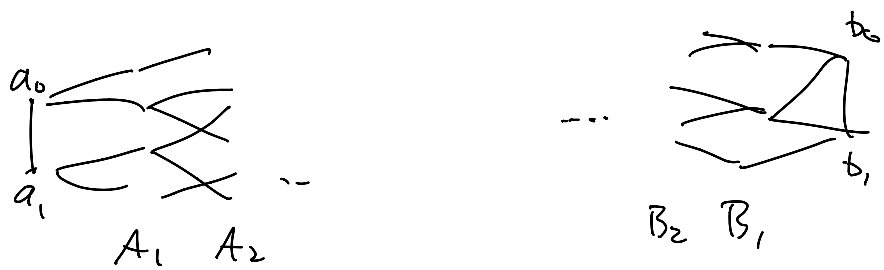
$$\lambda_2 \leq d - 2\sqrt{d-1} + \frac{2\sqrt{d-1} - 1}{k+1}$$

$A_0 = \{a_0, a_1\}$ $\quad A_i = \{a : dist(a, A_0) = i\}$

$B_0 = \{b_0, b_1\}$ $\quad B_i = \{a : dist(a, B_0) = i\}$

$A_0, \ldots, A_k$ disjoint, $B_0, \ldots B_k$ disjoint

no edges from A_i to B_j, any $i, j \leq k$



$x(A_0) = 1$

$x(A_i) = (d-1)^{-i/2}$

$x(B_0) = -\beta$

$x(B_i) = -\beta\,(d-1)^{i/2}$

choose β s.t. $1^T x = 0$

$$\frac{x^T L x}{x^T x} = \frac{A\text{-edges} + B\text{-edges}}{A\text{-verts} + B\text{-verts}} \le \max\left(\frac{A\text{-edges}}{A\text{-verts}}, \frac{B\text{-edges}}{B\text{-verts}}\right)$$

$$A\text{-verts} = \sum_{i=0}^{k} |A_i| (d-1)^{-i/2}$$

$$A\text{-edges} \le \sum_{i=0}^{k-1} |A_i| (d-1) \frac{\left(1 - \frac{1}{\sqrt{d-1}}\right)^2}{\left((d-1)^{-i/2}\right)^2} + |A_k| (d-1) \cdot (d-1) (d-1)^{-k}$$

$$= \sum_{i=0}^{k-1} \frac{|A_i|}{(d-1)^i} \left(d - 2\sqrt{d-1}\right)$$

$$(d-1)\left(1 - \frac{1}{\sqrt{d-1}}\right)^2$$

$$= \left(\sqrt{d-1} - 1\right)^2 = d - 1 - 2\sqrt{d-1} + 1$$

$$= d - 2\sqrt{d-1}$$

$$+ \frac{|A_k|}{(d-1)^k} (d-1)$$

$$= \underbrace{\sum_{i=0}^{k} \frac{|A_i|}{(d-1)^i} \left(d - 2\sqrt{d-1}\right)}_{\le \left(d - 2\sqrt{d-1}\right) A\text{-verts}} + \underbrace{\frac{|A_k|}{(d-1)^k} \left(2\sqrt{d-1} - 1\right)}$$

$$|A_k| \le (d-1)^{k-i} |A_i|$$

$$\Rightarrow |A_k| \le \frac{1}{k+1} \sum_{i=0}^{k} (d-1)^{k-i} |A_i|$$

$$\frac{|A_k|}{(d-1)^k} \left(2\sqrt{d-1} - 1\right) \le \frac{1}{k+1} \left(2\sqrt{d-1} - 1\right) \cdot \underbrace{\sum_{i=0}^{k} \frac{|A_i|}{(d-1)^i}}_{\le A\text{-verts}}$$