Rayleigh's Monotonicity theorem:

increasing resistances does not decrease effective resistance.

Do with effective spring constants $= \frac{1}{R_{eff}}$

Let $C_{eff}(s,t)$ = effective spring constant of $G=(V,E,w)$ between s and t

$$= \min_{\substack{x(s)=0 \\ x(t)=1}} x^T L x$$

Let $\hat{G}=(V,E,\hat{w})$ satisfy $\hat{w}_{a,b} \le w_{a,b}$ for all $(a,b) \in E$.

Let $\hat{C}_{eff}(s,t)$ = version for $\hat{G}$. $\hat{L}$ Lap of $\hat{G}$.

<u>Thm</u> $\forall s,t \quad \hat{C}_{eff}(s,t) \le C_{eff}(s,t)$

<u>proof</u>

$$C_{eff}(s,t) = \min_{\substack{x(s)=0 \\ x(t)=1}} x^T L x.$$

Let x be vector on which minimum is achieved.

Then $$x^T L x \ge x^T \hat{L} x \ge \min_{\substack{\hat{x} \text{ s.t.} \\ \hat{x}(s)=0 \\ \hat{x}(t)=1}} \hat{x}^T \hat{L} \hat{x} = \hat{C}_{eff}(s,t).$$

Demo?

For an edge (a,b), $R_{eff}(a,b)$ is a measure of its importance. Will later sample edges with prob proportional to $R_{eff}(a,b)$.

Let's see how to compute and store estimates efficiently.

Recall $R_{eff}(a,b) = (\delta_a - \delta_b)^T L^+ (\delta_a - \delta_b)$

$$= \| L^{+/2} (\delta_a - \delta_b) \|^2$$

$$= \| L^{+/2} \delta_a - L^{+/2} \delta_b \|^2$$

$$= dist(L^{+/2}\delta_a, L^{+/2}\delta_b)^2$$

Is square of a Euclidean distance.

Johnson-Lindenstrauss:

A Euclidean distance on n vectors $x_1 \ldots x_n$ is well-approximated by a distance in $O(\lg n)$ dimensions.

Will use Gaussian random variables.

Thm Let $x_1, \ldots, x_n \in \mathbb{R}^k$, $\varepsilon > 0$, $\delta > 0$. $d = \frac{8 \ln(n^2/\delta)}{\varepsilon^2}$.

Let R be random d-by-k matrix of independent $N(0, 1/d)$ variables.

Then with prob $\geq 1-\delta$, $\forall a \neq b$

$$(1-\varepsilon) dist(x_a, x_b) \leq dist(Rx_a, Rx_b) \leq (1+\varepsilon) dist(x_a, x_b)$$

So, instead of storing $x_a = L^{+/2}\delta_a \in \mathbb{R}^n$

store $y_a = R L^{+/2}\delta_a \in \mathbb{R}^d$

Computing $(\delta_a - \delta_b)^T L^+ (\delta_a - \delta_b)$ requires solving a system in L.

Note can do this in time essentially $O(m\sqrt{\lg n})$.

Will see how to approx all $\mathrm{Reff}(a,b)$ using only $O(\lg n)$ such solves.

Recall $L = UWU^T$ where $U \in \mathbb{R}^{n\times m}$ signed edge-vertex

$W \in \mathbb{R}^{m\times m}$ diagonal of weights

We have $L^+ = L^+ L L^+ = L^+ U W U^T L^+$

$$= L^+ U W^{1/2} \cdot W^{1/2} U^T L^+$$

so, $(\delta_a - \delta_b) L^+ (\delta_a - \delta_b) = \| W^{1/2} U^T L^+ (\delta_a - \delta_b) \|^2$

$$= \mathrm{dist}(M\delta_a, M\delta_b)^2$$

for $M = W^{1/2} U^T L^+$

Choose $R \in \mathbb{R}^{d\times m}$ $N(0, 1/\sqrt{d})$, and set

$y_a = R M \delta_a \in \mathbb{R}^d$

By JL, probably $\mathrm{dist}(y_a, y_b)^2 \in (1\pm\varepsilon)\,\mathrm{Reff}(a,b)$, $\forall a,b$.

To compute, multiply each of d rows of R
by $\underbrace{W^{1/2}, U^T}_{O(m) \text{ entries}}$ and then $\underbrace{L^+}_{\text{can do quickly}}$

Gaussian Random Variables

$N(0,1)$ has density $p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

If $X = \sum_{i=1}^{n} r_i$ $\quad r_i \in \pm 1$ prob $1/2$

$$\Pr\left[\frac{X}{\sqrt{n}} \in [a,b] \right] \to \int_a^b p(x)\,dx$$

$N(0,\sigma^2)$ has density $\frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$

multiply an $N(0,1)$ by σ.

Claim If $r_1 \dots r_n$ are indep $r_i \leftarrow N(0,\sigma_i^2)$,
then $\sum r_i$ is $N(0, \sum_i \sigma_i^2)$

Variances of sums ~~of~~ indep vars add.
This says is Gaussian.

For $x \in \mathbb{R}^d$ and r a vector of indep $N(0,1)$ vars,
$x^T r$ is $N(0, \|x\|^2)$

because $x^T r = \sum_i \underbrace{x(i) r(i)}_{\text{is } N(0, x(i)^2)}$

If r is vector of indep $N(0, \sigma^2)$ vars,

$x^T r$ is $N(0, \sigma^2 \|x\|^2)$.

---

let r be a vector of d indep $N(0,1)$ vars.

$\|r\|^2$ is called a χ^2 random variable.

<u>Thm</u> For $\varepsilon < 1$,

$$\Pr\left[\,\left|\, \|r\|^2 - d \,\right| > \varepsilon d\,\right] \le 2 \exp\left(-\varepsilon^2 d / 8\right)$$

Now, to prove JL.

<u>Thm</u> let $x_1, \ldots, x_n \in \mathbb{R}^k$, $\varepsilon > 0$, $\delta > 0$. $d = \dfrac{8 \ln(n^2/\delta)}{\varepsilon^2}$.

let R be random d-by-k matrix of independent $N(0, 1/d)$ variables.

Then with prob $\ge 1 - \delta$, $\forall\, a \ne b$

$$(1-\varepsilon)\, \mathrm{dist}(x_a, x_b) \le \mathrm{dist}(Rx_a, Rx_b) \le (1+\varepsilon)\, \mathrm{dist}(x_a, x_b) \quad (*)$$

<u>proof</u> for $a \neq b$, $(*)$ fails when

$$\Big| \|Rx_a - Rx_b\|^2 - \|x_a - x_b\|^2 \Big| > \varepsilon \|x_a - x_b\|^2$$

$$\|Rx_a - Rx_b\|^2 = \|R(x_a - x_b)\|^2.$$

Each entry of $R(x_a - x_b)$ is $N\left(0, \frac{\|x_a - x_b\|^2}{d}\right)$

Translating theorem says

$$\Pr\Big[\big| \|R(x_a - x_b)\|^2 - \|x_a - x_b\|^2 \big| > \varepsilon \|x_a - x_b\|^2 \Big] \le 2\exp\left(\frac{-d\varepsilon^2}{8}\right)$$

$$= 2\exp\left(-\ln\left(n^2/\delta\right)\right) = \frac{2\delta}{n^2}$$

So, prob $\exists a, b$ that violate $(*)$

$$\le \binom{n}{2} \frac{2\delta}{n^2} < \delta.$$