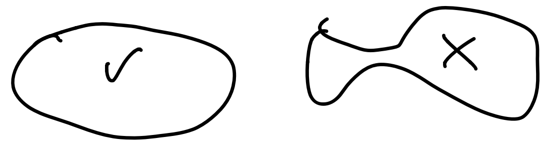
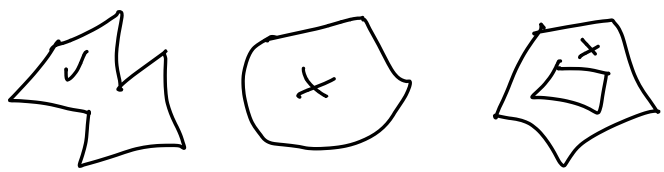
Convex Polygon.

$C \subseteq \mathbb{R}^2$ is convex if $\forall\, x, y \in C$, segment $xy \subseteq C$

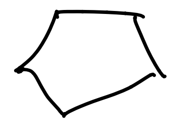


$C$ is polygon if connected

boundary is finite # of line segments



Convex Polygon => Convex and polygon

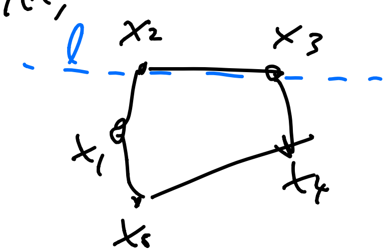


$x_1, \dots, x_k$ are corners of a strictly convex polygon if

segments $(x_i, x_{i+1})$, $(x_1, x_k)$ enclose a convex polygon

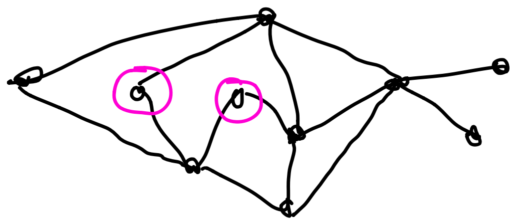
and all angles $< 180^\circ$

equiv: for every edge $(x_i, x_{i+1})$ and $\ell$ the line through $x_i$ and $x_{i+1}$, all others on one side of $\ell$



$G = (V, E)$ is planar if $\exists\, z : V \to \mathbb{R}^2$

s.t. line segments $\overline{z(a), z(b)}$ $(a,b) \in E$ do not intersect.



$G$ is $k$-connected if $G(V-S)$ is connected

for all $|S| < k$.

If consider $\mathbb{R}^2 - \{ \overline{z(a)\, z(b)} \;\; (a,b) \in E \}$

obtain regions called Faces, boardered by edges

and vertices.

3 con & planar $\Rightarrow$ Faces are determined

are polygons

and are simple cycles in $G$.

every edge on two faces

Henceforth $G$ 3-con & Planar

Edge contractions: if $(a,b)$ an edge,
replace $a, b$ with vertex $ab$,
and edges $(ab, c)$ for all $(a,c) \in E$ and $(b,c) \in E$

Planar $\Rightarrow$ planar after contract.

Minor: contract and remove edges

$G$ planar iff $K_5$ and $K_{3,3}$ are not minors.

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Tutte: let $B$ be vertices on a face
let $z$ map $B$ to corners of a convex polygon
every vertex not in $B$ is average of
neighbors.
This embedding is planar. If draw edges
as straight lines, every face is a strictly
convex polygon.

Need to rule out $z(a) = z(b)$ - later

Claim 1. $\forall\, a \notin B$ lie strictly

Eliminate all vertices not in $B \cup \{a\}$.

Now find $z(a)$ is an average of

$z(b)$ for $b \in B$.

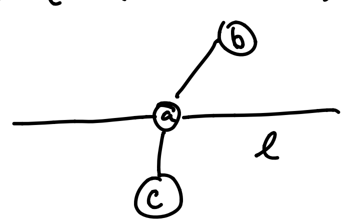
3-connected $\Rightarrow$ $a$ has $\geq 3$ neighbors in $B$

so strictly convex $\Rightarrow$ strictly inside

Claim 2. If $\ell$ is a line through $z(a)$

and $\exists\, b \sim a$ s.t. $z(b)$ on one side of $\ell$,

then is $c \sim a$ s.t. $z(c)$ is on other side



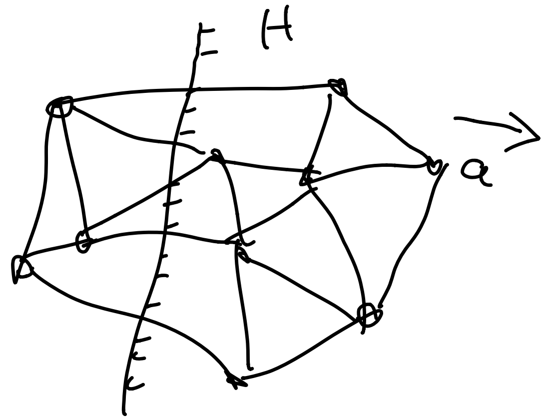

Lem 1 let $H$ be a halfspace: all points on a line

and one side of it. let $S = \{a : z(a) \in H\}$.

Then $G(S)$ is connected.

proof. let $H = \{z : t^T z \geq \mu\}$ and let

$a = \arg\max_a t^T z(a)$




For any $b \in S$ can find path to $a$ that stays in $S$. In fact $t^T z(c)$ nondecreasing on path.

If $c$ has neighbor increasing $t^T$, go to that

otherwise, consider any path to $a$.

It must have a first vertex with a neighbor not on line, and by Claim 2 a nbr above it.

<u>Lem 2</u> No vertex is colinear with all its neighbors.

<u>proof</u> let $l$ go through $a$, let $S^+$ and $S^-$ be above and below $l$.

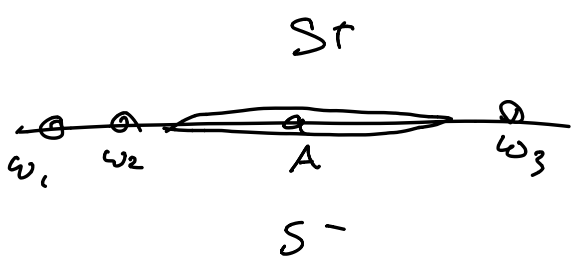
Assume bwoc all nbrs of $a$ on line.

Let $A$ = reachable from $a$ with all nbrs in $l$.

$a \in A$

Let $W$ = nbrs of $A$ on $\ell_1$ but not in $A$.

3-connected $\Rightarrow$ $|W| \ge 3$




Contracting $S^+$, $S^-$, $A$ gives a $K_{3,3}$ minor

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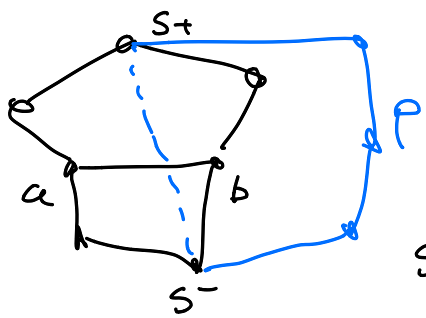
Lem 3

Let $(a,b)$ be an edge on faces with vertex sets $F^+$, $F^-$

Let $s^+ \in F^+ - \{a,b\}$ and $s^- \in F^- - \{a,b\}$ and let $P$ be a path from $s^+$ to $s^-$ that does not contain $a$ or $b$.

Then every path from $a$ to $b$ is $(a,b)$ or intersects $P$.

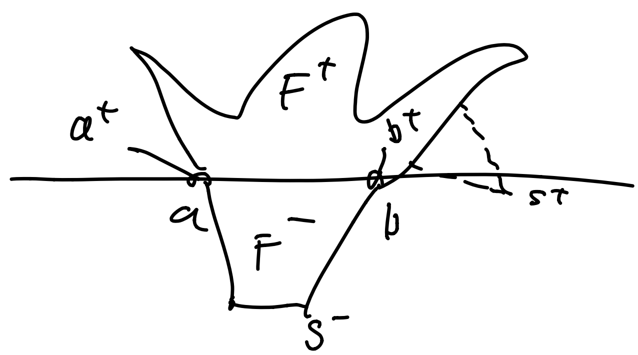
proof: in a planar drawing




Consider curve $P$ + path from $s^+$ to $s^-$ that only crosses $(a,b)$

Separates $a$ from $b$

Thm Let $(a,b)$ be any non-boundary edge, let $l$ be a line through $z(a)$ and $z(b)$, and let $F^+$ and $F^-$ be the faces with $(a,b)$. Then all vertices of $F^+$ and $F^-$ lie on opposite sides of $l$, and none lie on $l$.




proof Assume bwoc is a vertex $s^+ \in F^+$ and $s^- \in F^-$ both on or below.

As halfspace below line is connected, is a path $P^-$ from $s^+$ to $s^-$ strictly below line.

By claim 2, $a$ and $b$ have neighbors $a^+$ and $b^+$ above the line.

So, is a path $P^+$ above the line from $a^+$ to $b^+$

So, $a, a^+, P^+, b^+, b$ connects $a$ to $b$, is not $(a,b)$ and does not intersect $P^-$

Contradicts lem 3.

$\Rightarrow$ All faces are strictly convex polygons.