Recall $E|\overset{v}{B}$ $\quad L = B^T W B \quad i = WBv \quad i_{ext} = Lv$

$v = L^+ i_{ext}$.

<u>Effective resistance</u> $\quad U = IR \quad R = \frac{U}{I}$

$= \frac{\text{voltage difference}}{\text{current flow.}}$

Consider $i_{ext} = \delta_a - \delta_b$ flows 1 from a to b in graph

$$v = L^+ i_{ext} = L^+(\delta_a - \delta_b)$$

difference $= v(a) - v(b) = (\delta_a - \delta_b)^T v$

$$R_{eff}(a,b) = (\delta_a - \delta_b)^T L^+ (\delta_a - \delta_b)$$

Will see later that is a distance. For now,

(let $L^{+/2} = (L^+)^{1/2}$.

Every psd matrix A has $A^{1/2}$ s.t. $A^{1/2} \cdot A^{1/2} = A$

if $A = \sum \lambda_i \psi_i \psi_i^T$, $A^{1/2} = \sum_i \lambda_i^{1/2} \psi_i \psi_i^T$)

$$R_{eff}(a,b) = (\delta_a - \delta_b)^T L^{+/2} \cdot L^{+/2} (\delta_a - \delta_b)$$

$$= \| L^{+/2}(\delta_a - \delta_b) \|_2^2$$

$$= \| L^{+/2}\delta_a - L^{+/2}\delta_b \|_2^2$$

By energy minimization.

Recall in spring of constant w, $\mathcal{E} = \frac{1}{2} w l^2$

when stretch to length l.

So, let's stretch to length 1,

measure minimum energy, and double to get effective spring constant

$$\mathcal{E}(x) = \frac{1}{2} \sum_{a \sim b} w_{a,b} (x(a) - x(b))^2$$

Let's fix $x(s) = 1$, $x(t) = 0$

And want harmonic elsewhere.

Set
$$y = \frac{L^+ (\delta_s - \delta_t)}{R_{eff}(s,t)}$$

Gives
$$y(s) - y(t) = \frac{(\delta_s - \delta_t)^T L^+ (\delta_s - \delta_t)}{R_{eff}(s,t)} = 1$$

Could shift by $x = y - \mathbf{1}\, y(t)$

Now $x(t) = 0$, $x(s) = 1$

Harmonic on $V - \{s,t\} \Rightarrow$ minimizes energy

$$\mathcal{E}(x) = \tfrac{1}{2} x^T L x = \tfrac{1}{2} y^T L y$$

$$= \frac{1}{2} \frac{1}{\mathrm{Reff}(s,t)^2} (\delta_s - \delta_t)^T L^+ L L^+ (\delta_s - \delta_t)$$

$$= \frac{1}{2} \frac{1}{\mathrm{Reff}(s,t)^2} (\delta_s - \delta_t)^T L^+ (\delta_s - \delta_t)$$

$$= \frac{1}{2} \frac{1}{\mathrm{Reff}(s,t)}$$

So, effective spring constant $= \dfrac{1}{\mathrm{Reff}(s,t)}$

Classic examples

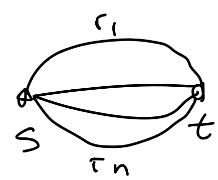
Path with n vertices, edges of resistance $r_1, \dots, r_{n-1}$

$\mathrm{Reff}(1,n) = r_1 + \dots + r_{n-1}$

proof set $v(a) = r_1 + \dots + r_{a-1}$ $\quad v(1) = 0, \quad v(n) = r_1 + \dots + r_{n-1}$

current over edge $(a-1, a)$ is $\dfrac{v(a-1) - v(a)}{r_{a-1}} = \dfrac{-r_{a-1}}{r_{a-1}} = 1$

So, corresponds to a flow of value 1 from n to 1.

Parallel edges 

Claim $\mathrm{Reff}(s,t) = \dfrac{1}{1/r_1 + \dots + 1/r_n}$

proof $R = \frac{V}{I}$ set $v(s) = 1$ $v(t) = 0$

flow on edge i is $\frac{(v(s) - v(t))}{r_i} = \frac{1}{r_i}$

So, total flow $= \sum_i \frac{1}{r_i}$

$\Rightarrow R_{eff}(s,t) = \frac{1}{\sum_i \frac{1}{r_i}}$

If view formula as $i = (v(s) - v(t)) w_{s,t}$

$w_i = 1/r_i$, then add weights of parallel edges.

Equivalent networks. Given B, want matrix L_B

s.t. $i_B = L_B v(B)$ when v harmonic on $S = V - B$.

To do it slowly, first consider $B = \{2, \ldots, n\}$ $S = \{1\}$

Let $N = \{a : a \sim 1\}$.

Want to compute Lv given $v(B)$ and $v(1) = \frac{1}{d(1)} \sum_{a \sim 1} w_{1,a} v(a)$

Substitute for $v(1)$ in

$i_{ext}(a) = d(a) v(a) - \sum_{b \sim a} w_{a,b} v(b)$ when $1 \sim a$

no change if $a \not\sim 1$

For $a \sim 1$ becomes

$$d(a)\,v(a) - \sum_{\substack{b\sim a \\ b\neq 1}} w_{a,b}\,v(b) - \frac{w_{a,1}}{d(1)} \sum_{c\sim 1} w_{1,c}\,v(c)$$

$$= v(a)\left[d(a) - \frac{w_{1,a}^2}{d(1)} \right] - \sum_{\substack{c\sim 1 \\ c\neq a}} \frac{w_{a,1}w_{1,c}}{d(1)} v(c) - \sum_{\substack{b\sim a \\ b\neq 1}} w_{a,b}\,v(b)$$

Claim is result of elimination on row/col 1

and, is a Laplacian equation on B.

We removed node 1 and attached edges, and

put back a clique on nbrs,

where for $a, c \sim 1$ have edge of wt $\frac{w_{a,1}w_{c,1}}{d(1)}$

proof is Laplacian

1. is symmetric: same charge to $\text{test}(c)$ in a

2. off diagonal terms negative

3. sum of coefficients is 0, as

$$w_{1,a} - \frac{w_{1,a}^2}{d(1)} - \sum_{\substack{c\sim 1 \\ c\neq a}} \frac{w_{1,c}w_{1,a}}{d(1)} = w_{1,a} - \frac{w_{1,a}}{d(1)} \sum_{c\sim 1} w_{1,c} = 0$$

In Energy term.

Compute L_B so that

$$v(B)^T L_B v(B) = \begin{pmatrix} \frac{1}{d(1)} \sum_{a \sim 1} w_{1,a} v(a) \\ v(B) \end{pmatrix}^T L \begin{pmatrix} \frac{1}{d(1)} \sum_{a \sim 1} w_{1,a} v(a) \\ v(B) \end{pmatrix}$$

idea: $\frac{1}{d(1)} \sum_{a \sim 1} w_{1,a} v(a) = - \frac{L(1,B) v(B)}{L(1,1)}$

substituting this in yields

$$v(B)^T \underbrace{\left[L(B,B) - \frac{L(B,1) L(1,B)}{L(1,1)} \right]}_{\substack{\parallel \\ L_B}} v(B)$$

To check Laplacian, note: only decrease entries,

and $L_B \mathbb{1}_B = 0$ proof: gives $v(1) = 1$, so $v^T L v = 0$

Or, $\mathbb{1}_B^T L(B,B) \mathbb{1}_B = d(1)$ and $L(1,B) \mathbb{1}_B = d(1)$

$L(1,1) = d(1)$.

Is what get by GE on row/col 1.

Eliminating many vertices at once:

Does not depend on order!

To elim entries in row $a \in B$ and cols in S
using rows in S, mult by coeffs c so that

$$L(a,S) - c L(S,S) = 0$$

So, $c = L(a,S) L(S,S)^{-1}$

giving $L_B(a,:) = L(a,:) - L(a,S) L(S,S)^{-1} L(a,:)$

restricting to rows and cols in B we get

$$L_B(B,B) = L(B,B) - L(B,S) L(S,S)^{-1} L(S,B)$$

is Schur complement with respect to S
or onto B

To show equiv of harmonic on S:

harmonic $\Rightarrow$ $L(S,S) v(S) + L(S,B) v(B) = 0$

$$v(S) = - L(S,S)^{-1} L(S,B) v(B)$$

$$i_{ext}(B) = L(B,S) v(S) + L(B,B) v(B)$$

$$= \left[L(B,B) - L(B,S) L(S,S)^{-1} L(S,B) \right] v_B$$

If $B = \{s, t\}$ get down to one edge,
whose weight is $\frac{1}{R_{eff}(s,t)}$

In particular, for $s, t \in B$,

$$(\delta_s - \delta_t)^T L_B^+ (\delta_s - \delta_t) = (\delta_s - \delta_t)^T L^+ (\delta_s - \delta_t)$$

Is how GE works to solve $i_{ext} = Lv$

Order vertices $1, \dots, n$

Construct $L_{\{a, \dots n\}}$ for each a.

Given $i_{ext}(a)$ and $v(a+1), \dots, v(n)$
solve for $v(a)$.

If $i_{ext}(a) = 0$, $v(a) = \frac{1}{d(a)} \sum_{b \sim a} w_{a,b} v(b)$

o/w need to account for $i_{ext}(a)$

Reff as distance:

assert $\forall a, b, c$ $R_{eff}(a,b) + R_{eff}(b,c) \geq R_{eff}(a,c)$

Only need to prove for 3-node graphs.