# Physical models and Harmonic Functions on Graphs

$G = (V, E, w)$ connected. $B \subset V$, $B \neq \emptyset$. Boundary

$S = V - B$.

$x : V \to \mathbb{R}$ is harmonic on $S$ if

$\forall a \in S \quad x(a) = \frac{1}{d(a)} \sum_{b \sim a} w_{a,b}\, x(b)$

weighted average of its neighbors.

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<u>Examples</u> Random walk. Distinguish $s, t \in V$

$B = \{s, t\}$. Consider a random walk that stops when it hits $s$ or $t$.

$x(a) = \Pr[\text{walk that starts at } a \text{ stops at } s]$

$x(s) = 1$, $x(t) = 0$. because stops there.

Claim $x$ is harmonic on $S$.

For $a \notin \{s,t\}$

Prob stops at $t$ $= \sum_{b \sim a} \text{prob}[\text{moves to } b] \cdot \text{prob}(b \to t)$

$$x(a) = \sum_{b \sim a} \frac{w_{a,b}}{d(a)} x(b)$$

Consider path on $\{1, \dots, n\}$ $\quad t = 1, \quad s = n$

$$x(a) = \frac{a-1}{n-1}$$

$1 — 2 — 3 — 4$

$0 \quad \tfrac{1}{3} \quad \tfrac{2}{3} \quad 1$

Will show that solution to these equations is unique, given values of $x$ on $B$.

First, more intuition

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View each edge as a spring. $w_{a,b}$ = spring constant

higher = stronger connection.

Fix values on boundary $B$, let Hooke's law determine the rest.

Hooke: force spring $(a,b)$ exerts on $a$ is $(x(b) - x(a)) w_{a,b}$

Equilibrium: all forces zero, except on $B$.

$$\sum_{b \sim a} (x(b) - x(a)) w_{a,b} = 0 \iff \sum_{b \sim a} x(b) w_{a,b} = d(a) x(a)$$

$$x(a) = \frac{1}{d(a)} \sum_{b \sim a} w_{a,b} x(b)$$

Harmonic on $S$

Path again: if $x(1) = 1$ $\quad x(n) = n$,

must have $x(a) = a$

Uniqueness of solutions, and how to find them.

Harmonic equation is $a \in S \Rightarrow d(a)x(a) - \sum_{b \sim a} w_{a,b} x(b) = 0$

That is $\delta_a^T L x = 0$, a row of $L$, for each $a \in S$.

Move $x(b)$ for $b \in B$ to rhs.

$$d(a)x(a) - \sum_{\substack{b \sim a \\ b \in S}} w_{a,b} x(b) = \sum_{\substack{b \sim a \\ b \in B}} x(b)$$

Today:

$x(S)$ = sub-vector, not sum over $S$.

Becomes $L(S,S)\,x(S) = M(S,B)\,x(B)$

$$\text{So, } x(S) = L(S,S)^{-1} M(S,B)\,x(B)$$

Need to show $L(S,S)$ exists. True if $G$ connected & $B \neq \emptyset$.

Because $L(S,S)$ is nice.

Claim $L(S,S) = L_{G(S)} + X_S$

where $X_S$ diagonal $X_S(a,a) = \sum_{\substack{b \sim a \\ b \in B}} w_{a,b}$

Lem. Let $H$ be connected and $X$ be non-neg, nonzero diagonal. Then $L_H + X$ is pos def.

proof need to show $\forall x \neq 0$ $x^T(L_H + X)x > 0$

If $x$ is non-constant $x^T L_H x > 0$.

If $x$ is constant $= c\mathbf{1}$, $c \neq 0$ so

$$x^T X x = c^2 \sum_a X(a,a) > 0$$

As $L_H$ and $X$ psd, $x^T(L_H + X)x \geq \min(x^T L_H x, x^T X x) > 0$.

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Almost proves $L(S,S)$ pos def, but $G(S)$ could be disconnected.

Lem If $G$ connected, $B \neq \emptyset$, $S = V - B$, then $L(S,S)$ is pos def.

Let $S_1, \ldots S_k$ be connected components of $G(S)$.

Then $L(S,S)$ has form

$$\begin{pmatrix} L(S_1,S_1) & & & \\ & L(S_2,S_2) & & \\ & & \ddots & \\ & & & L(S_k,S_k) \end{pmatrix}$$

Each $G(S_i)$ is connected.

And, $\exists a \in S_i$ s.t. $X(a,a) > 0$, because $G \Rightarrow$ is an edge of $G$ connecting some vertex of $S_i$ to $B$.

$$L(S,S) = \bigoplus_{i=1}^{k} \left( L(S_i,S_i) + X(S_i,S_i) \right) \quad \text{each pos def.}$$

Energy - last term to identify.

Energy in spring with const $w$ when stretched to length $l$ is $\frac{1}{2} w l^2$

So, energy in network is

$$\frac{1}{2} \sum_{(a,b) \in E} w_{a,b} \left(x(a) - x(b)\right)^2 = \frac{1}{2} x^T L x$$

Physics says energy minimized at equilibrium,

so, takes $\frac{\partial}{\partial x(a)} \frac{1}{2} x^T L x = 0$

$$\frac{\partial}{\partial x(a)} \frac{1}{2} \sum_{b \sim a} w_{a,b} \left(x(a) - x(b)\right)^2 = \frac{1}{2} \sum_{b \sim a} w_{a,b}\, 2\left(x(a) - x(b)\right)$$

$$= \sum_{b \sim a} w_{a,b} \left(x(a) - x(b)\right) = 0$$

$\iff$ $x$ harmonic at $a$

## Resistor Networks

Resistance of edge $a,b$ is $r_{a,b} = \frac{1}{w_{a,b}}$

Associate voltages with vertices, and flows on edges.

Ohm's law: $V = I R$

voltage difference — current — resistance

$i(a,b) =$ current flow from $a$ to $b$

$i(b,a) = -i(a,b)$

$v(a) - v(b) = i(a,b)\, r_{a,b}$

$i(a,b) = w_{a,b}\,(v(a) - v(b))$ current flows high to low.

$U =$ signed edge-vertex adj. mat is $E \times V$

$$U((a,b),c) = \begin{cases} 1 & a=c \\ -1 & b=c \\ 0 & \text{o.w.} \end{cases}$$

are picking an arbitrary orientation for each edge.

$W = E \times E$ diagonal edge weight matrix.

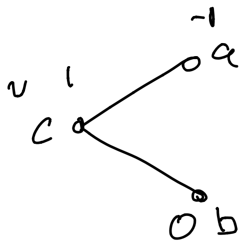
$i = W U v$

$i_{ext} \triangleq$ current entering. $i_{ext}(a) =$ current entering at $a$.

$i_{ext}(a) = \sum_{b \sim a} i(a,b)$ no current stored at a node.

$i_{ext} = U^T i$

Check signs :




$$U: \begin{array}{c|ccc} & a & b & c \\ \hline c,a & -1 & 0 & 1 \\ b,c & 0 & 1 & -1 \end{array}$$

$$Uv = \begin{matrix} 2 \\ -1 \end{matrix}$$

$i(c,a) = 2$

$i(b,c) = -1$

$$U^T i = \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{matrix} -2 & i_{ext}(a) \\ -1 & i_{ext}(b) \\ 3 & i_{ext}(c) \end{matrix}$$

$$L = U^T W U \qquad i_{ext} = Lv$$

$$= \sum_{(a,b) \in E} w_{a,b} (\delta_a - \delta_b)(\delta_a - \delta_b)^T$$

$$B = \{ a : i_{ext}(a) \neq 0 \}$$

For $a \in S$, $\quad i_{ext}(a) = 0$

$$\Rightarrow \delta_a^T L v = 0$$

$$\Rightarrow d(a) v(a) = \sum_{b \sim a} w_{a,b} v(b)$$

$v$ is harmonic at $a$.

As $i(a,b) = w_{a,b} (v(a) - v(b))$

$$\Rightarrow \sum_{b \sim a} i(a,b) = \sum_{b \sim a} w_{a,b} (v(a) - v(b)) = 0$$

$\Rightarrow$ zero net flow at $a$.

Given $i_{ext}$, how solve for $v$?

$i_{ext} = Lv$, so $v = L^{-1} i_{ext}$

But there is no $L^{-1}$ !?

There is a solution if $\sum_a i_{ext}(a) = 0$, and $G$ connected.

Called the pseudo-inverse $L^+$

$$L^+ L = L L^+ = I - \tfrac{1}{n} \mathbf{1}\mathbf{1}^T = \tfrac{1}{n} L_{K_n}$$

$$= \text{projection onto span}(L)$$

$L^+$ is identity on the span.

If $L = \sum_i \lambda_i \psi_i \psi_i^T$, $L^+ = \sum_{i : \lambda_i \neq 0} \frac{1}{\lambda_i} \psi_i \psi_i^T$