If $\exists\, S \subseteq V$ s.t. $d(S) < d(V)/32$, $\phi(S)$ small,

find small T with $\phi(T)$ small.

By setting $p_0 = \delta_a$, and running Cheeger on

$y = D^{-1} p_t$ for $t = 1/2\phi(S)$, for some $a \in S$

Input is a, target s, target ϕ

alg works for some $a \in S$ if $d(S) \le s$, $\phi(S) \le \phi$,

$s \le d(V)/32$, $\phi(S) \le \frac{1}{\ln 8|S|}$

Assume $\phi(S) \le \phi$, $t = \frac{1}{2\phi}$.

First, show is a vertex $a \in S$ st.

$$1_S^T \tilde{W}^t \delta_a \ge \frac{1}{2}$$

<u>Lem</u> Let $p_S(a) = \frac{d(a)}{d(S)}$ $a \in S$

0 o.w.

Then $1_{V-S}^T \tilde{W}^t p_S \le t\phi(S)/2 \le \frac{1}{4}$

<u>proof</u> Set $p_i = \tilde{W}^i p_S$. In each step prob walk leaves S

$\le \phi(S)/2$. In first step, prob is

$$\sum_{a \in S} p_S(a) \cdot \frac{1}{2} \cdot \sum_{\substack{b \sim a \\ b \notin S}} \frac{w_{a,b}}{d(a)} = \frac{1}{2} \frac{1}{d(S)} \sum_{\substack{a \in S \\ b \notin S}} w_{a,b} = \frac{1}{2} \frac{w(\partial(S))}{d(S)} = \frac{1}{2}\phi(S).$$

To argue for later steps, prove $p_i(a) \le p_S(a)$ $\forall a \in S$.

in fact prove $p_i(a) \le \frac{d(a)}{d(S)}$ $\forall a$

Holds for $i=0$. By induction

$$\delta_a^T p_i = \delta_a^T \tilde{W} p_{i-1} \le \delta_a^T \tilde{W} \frac{d}{d(S)} = \delta_a^T \frac{d}{d(S)} = \frac{d(a)}{d(S)} \checkmark$$

<u>Def</u> a is good for S if $1_S^T \tilde{W}^t \delta_a \ge \frac{1}{2}$ and $\frac{d(a)}{d(S)} \ge \frac{1}{2|S|}$

<u>Lemma</u> There is an $a \in S$ that is good for S.

$$\sum_{a \in S} \frac{d(a)}{d(S)} 1_{V-S}^T \tilde{W}^t \delta_a \le \frac{1}{4}$$

let $B_1 = \{a \in S : 1_{V-S}^T \tilde{W}^t \delta_a > \frac{1}{2}\}$ $B_2 = \{a \in S : \frac{d(a)}{d(S)} < \frac{1}{2|S|}\}$

a is good if is in $S - B_1 - B_2$.

because $1_S^T \tilde{W}^t \delta_a = 1 - 1_{V-S}^T \tilde{W} \delta_a$

$$\sum_{a \in B_1} \frac{d(a)}{d(S)} < \frac{1}{2} \qquad \sum_{a \in B_2} \frac{d(a)}{d(S)} < \sum_{a \in B_2} \frac{1}{2|S|} \le \frac{1}{2}$$

so, $\exists$ good a.

Cheeger depends on $\frac{y^T L y}{y^T D y}$ where $y^T d$

$y = D^{-1} p$ means $\frac{p_t D^{-1} L D^{-1} p_t}{p_t D^{-1} p_t}$

<u>Claim</u> a good for $S \Rightarrow p_t^T D^{-1} p_t \geq \frac{1}{4 d(S)}$

<u>proof</u> $\frac{1}{2} \leq 1_S^T p_t = \sum_{a \in S} p_t(a) = \sum_{a \in S} \sqrt{d(a)}\; p_t(a) / \sqrt{d(a)}$

so, by Cauchy-Schwartz

$$\frac{1}{4} \leq d(S) \sum_{a \in S} \frac{p_t(a)^2}{d(a)} \leq d(S) \sum_{a} \frac{p_t(a)^2}{d(a)} =$$

$$d(S)\; p_t^T D^{-1} p_t$$

<u>Lem</u> let $p_t = \tilde{W}^t p_0$. Then

$$(\ast) = \frac{p_t^T D^{-1} L D^{-1} p_t}{p_t^T D^{-1} p_t} \leq 2 - 2 \left(\frac{p_t D^{-1} p_t}{p_0 D^{-1} p_0} \right)^{1/2t}$$

$p_0^T D^{-1} p_0 = \delta_a^T D^{-1} \delta_a = \frac{1}{d(a)}$. So, $\frac{p_t^T D^{-1} p_t}{p_0^T D^{-1} p_0} \geq \frac{d(a)}{4d(s)} \geq \frac{1}{8|S|}$

$$\left(\frac{1}{8|S|}\right)^{1/2t} = \exp\left(-\ln(8|S|) \,/2t\right)$$

$$\leq 1 - \ln(8|S|)/4t$$

if $\ln(8|S|) \leq \frac{1}{4t} \leq \ln(8|S|)\,\phi/2$

$\Longrightarrow$ $(\ast) \leq \ln(8|S|)\,\phi$

Use Power Means Inequality: For $\sum_i \omega_i = 1$, $\omega_i \geq 0$, $\lambda_i \geq 0$

$k > h$

$$\left(\sum \omega_i \lambda_i^k\right)^{1/k} \geq \left(\sum \omega_i \lambda_i^h\right)^{1/h}$$

Define $z_t = D^{-1/2} p_t$, so $(\ast) = \frac{z_t^T N z_t}{z_t^T z_t}$

In eigenbasis of N, $z_0 = \sum_i c_i \psi_i$

set $\gamma = \frac{1}{\sum c_i^2}$, so $\sum_i \gamma c_i^2 = 1$

$z_t = \left(D^{-1/2} \tilde{W} D^{1/2}\right)^t z_0$ $\quad D^{-1/2} \tilde{W} D^{1/2} = I - \frac{1}{2} N$,

eigs ω_i $\quad$ eigs ν_i

$$\nu_i = 2 - 2\omega_i$$

$$\gamma\, z_t^T z_t = \gamma \sum_i c_i^2 \omega_i^{2t}$$

$$\gamma z_t^T N z_t = \gamma \sum_i c_i^2 \omega_i^{2t} v_i = \gamma \sum_i c_i^2 \omega_i^{2t} (2 - 2\omega_i)$$

$$\frac{\gamma z_t^T N z_t}{\gamma z_t^T z_t} = \frac{2\gamma \sum_i c_i^2 \omega_i^{2t} - 2\gamma \sum_i c_i^2 \omega_i^{2t+1}}{\gamma \sum_i c_i^2 \omega_i^{2t}}$$

$$= 2 - 2 \underbrace{\frac{\sum_i \gamma c_i^2 \omega_i^{2t+1}}{\sum_i \gamma c_i^2 \omega_i^{2t}}}$$

$$\leq \left(\sum_i \gamma c_i^2 \omega_i^{2t} \right)^{1/2t}$$

$$= \left(\frac{\sum_i c_i^2 \omega_i^{2t}}{\sum_i c_i^2} \right)^{1/2t}$$

$$= \left(\frac{z_t^T z_t}{z_0^T z_0} \right)^{1/2t}$$

$$=$$

So, setting $y = D^{-1} p_t$ gives

$$\frac{y^T L y}{y^T D y} \leq \phi \cdot \ln 8|S|$$

Want to apply Cheeger. But, how make sure cut is small?

Work with $x(a) = \max(0, \cancel{y}(a) - \frac{1}{16s})$

$x^T L x \leq \cancel{x}^T L y$

is balanced, and mostly zero.

$$\sum_{a : x(a) > 0} d(a) \leq \frac{d(V)}{2}, \text{ because } s \leq \frac{d(V)}{32}$$

Lem For good a, $x^T D x \geq \frac{1}{2} y^T D y$.

Because $y^T D y \geq \frac{1}{4s}$ and

$\forall y \max\left(0, y - \frac{1}{16s}\right)^2 \geq y^2 - \frac{y}{8s}$

So, $$\sum_a \max\left(0, y(a) - \frac{1}{16s}\right)^2 \cdot d(a) \geq \sum_a y(a)^2 d(a) - \sum_a \frac{y(a) d(a)}{8s}$$

$$= y^T D y - \frac{1}{8s}$$

$$\geq \frac{1}{2} y^T D y$$

Now, $\frac{x^T L x}{x^T D x} \leq 2 \ln 8|S| \cdot \phi$

So, cheeger gives T s.t.

$$\phi(T) \leq \sqrt{4 \ln 8|S| \phi}, \quad d(T) \leq 16\, d(S)$$