A random walk on $G=(V,E)$ is a process with discrete time. Today all G connected.

If at vertex a at time t, at time $t+1$ $\Pr[\text{at } b] = \frac{w_{a,b}}{d(a)}$

Usually consider distribution instead of precise location.

Def p is a prob vector if $p(a) \geq 0 \ \forall a$ and $1^T p = 1$

let p_t be vec at time t. Usually $p_0 = \delta_a$ for some a.

Then $p_1 = \sum_{b \sim a} \frac{w_{a,b}}{d(a)} \delta_b = M D^{-1} p_0$

let $W = MD^{-1}$ be the <u>walk matrix</u> then

$$p_{t+1} = W p_t = W^t p_0$$

$$p_{t+1}(b) = \delta_b^T p_{t+1} = \delta_b^T M D^{-1} p_t =$$

$$= \delta_b^T M \sum_a p_t(a) \frac{\delta_a}{d(a)} = \sum_{a \sim b} \frac{w_{a,b}}{d(a)}$$

Usually consider lazy random walk

$$p_{t+1} = \tfrac{1}{2} p_t + \tfrac{1}{2} W p_t = \widehat{W} p_t$$

where $\widehat{W} \triangleq \tfrac{1}{2} I + \tfrac{1}{2} W$

Not symmetric, but

$D^{-1/2} W D^{1/2} = A \triangleq D^{-1/2} M D^{-1/2}$ is

So can get eigvecs and eigvals of W from A.

Claim $A\psi = \omega\psi \iff W(D^{1/2}\psi) = \omega(D^{1/2}\psi)$

$A = D^{-1/2} W D^{1/2}$

$D^{1/2} A = W D^{1/2}$

$D^{1/2} A \psi = W(D^{1/2}\psi) = \omega(D^{1/2}\psi)$

$\widetilde{W}$ same eigvecs.

Perron vec of W is d:

$M D^{-1} d = M\mathbb{1} = d$. has eigval 1.

So, eigvals W are in $[-1,1]$

$\Rightarrow$ eigvals of $\widetilde{W}$ are in $[0,1]$.

Let $1 = \omega_1 > \omega_2 \geq \cdots \geq \omega_n \geq 0$ be eigvals of $\widetilde{W}$

$D^{-1/2} d = d^{1/2}$ is a Perron vec of A.

$\psi_1 = \dfrac{d^{1/2}}{\|d^{1/2}\|}$ is the unit-norm version

Stable distribution $\pi = \frac{d}{d(V)}$ is a prob vector.

$\tilde{W} \cdot \pi = \pi$ If G connected

<u>Lem</u> For all p_0, $\tilde{W}^t p_0 \to \pi$.

<u>proof</u> let $\psi_1, \dots, \psi_n$ be eigvecs of A.

Write $D^{-1/2} p_0 = \sum_i c_i \psi_i$, $c_i = \psi_i^T D^{-1/2} p_0$

note $c_1 = \psi_1^T D^{-1/2} p_0 = \frac{d^{1/2}}{\|d^{1/2}\|} D^{-1/2} p_0 = \frac{1^T p_0}{\|d^{1/2}\|} = \frac{1}{\|d^{1/2}\|}$

$$
\begin{aligned}
p_t = \tilde{W}^t p_0 &= D^{1/2} \cdot D^{-1/2} \tilde{W}^t D^{1/2} \cdot D^{-1/2} p_0 \\
&= D^{1/2} \left(D^{-1/2} \tilde{W} D^{1/2} \right)^t \cdot D^{-1/2} p_0 \\
&= D^{1/2} \left(\tfrac{1}{2} I + \tfrac{1}{2} A \right)^t \sum_i c_i \psi_i \\
&= D^{1/2} \sum_i c_i \omega_i^t \psi_i \\
&= D^{1/2} c_1 \psi_1 + \underbrace{D^{1/2} \sum_{i>1} c_i \omega_i^t \psi_i}_{\to 0 \text{ as } \omega_i < 1}
\end{aligned}
$$

$c_1 D^{1/2} \psi_1 = \frac{1}{\|d^{1/2}\|} D^{1/2} \frac{d^{1/2}}{\|d^{1/2}\|} = \frac{d}{\|d^{1/2}\|^2} = \frac{d}{\sum d(a)} = \pi$

Thm For all a, b and t, if $p_0 = \delta_a$ then

$$\left| p_t(b) - \pi(b) \right| \le \sqrt{\frac{d(b)}{d(a)}}\ \omega_2^t$$

proof We need to show $\left| \delta_b^T \sum_{i>1} c_i \omega_i^t D^{1/2} \psi_i \right| \le \sqrt{\frac{d(b)}{d(a)}}\, \omega_2^t$

note $c_i = \psi_i^T D^{-1/2} \delta_a = \frac{1}{\sqrt{d(a)}} \psi_i^T \delta_a$

So $\delta_b^T \sum_{i>1} c_i \omega_i^t D^{1/2} \psi_i = \sqrt{\frac{d(b)}{d(a)}} \sum_{i>1} \omega_i^t \delta_b^T \psi_i \psi_i^T \delta_a$

$$\left| \sum_{i>1} \omega_i^t \delta_b^T \psi_i \psi_i^T \delta_a \right| \le \sum_{i\ge 2} \omega_i^t \left| \delta_b^T \psi_i \right| \left| \psi_i^T \delta_a \right|$$

$$\le \omega_2^t \sum_{i>1} \left| \delta_b^T \psi_i \right| \left| \psi_i^T \delta_a \right|$$

$$\le \omega_2^t \sum_{i\ge 1} \left| \delta_b^T \psi_i \right| \left| \psi_i^T \delta_a \right|$$

$$\le \omega_2^t \sqrt{\sum_{i\ge 1} (\delta_b^T \psi_i)^2} \sqrt{\sum_{i\ge 1} (\delta_a^T \psi_i)^2}$$

$$= \omega_2^t \, \|\delta_b\| \cdot \|\delta_a\|$$

$$= \omega_2^t$$

Normalized Laplacian

$$N = D^{-1/2} L D^{-1/2} = I - D^{-1/2} M D^{-1/2} = I - A$$

eigvals $0 = \nu_1 \le \nu_2 \le \dots \le \nu_n$

$\omega_i = 1 - \nu_i/2$

$$W = I - \tfrac{1}{2} D^{1/2} N D^{-1/2}$$

Proved $|P_t(b) - \pi(b)| \le \sqrt{\frac{d(b)}{d(a)}} (1 - \nu_2/2)^t$

Say walk has mixed if $|P_t(b) - \pi(b)| \le \frac{\pi(b)}{2}$

$$\sqrt{\frac{d(b)}{d(a)}} (1 - \nu_2/2)^t \le \frac{d(b)}{2d(V)}$$

$$\Longleftrightarrow \quad (1 - \nu_2/2)^t \le \frac{\sqrt{d(b)d(a)}}{2d(V)}$$

$$\leq$$

$$\exp(-t\nu_2/2)$$

So, want $t = 2 \ln\left(\frac{2d(V)}{\sqrt{d(a)d(b)}}\right) / \nu_2$

if $d(a) \approx d(b)$ $\forall a,b$, is

$$t \gtrsim 2 \ln(2n) / \nu_2$$

Sometimes only need $t \ge c/\nu_2$

To estimate ν_2, note

$$\frac{\lambda_2}{d_{max}} \le \nu_2 \le \frac{\lambda_2}{d_{min}}$$

Path: $\lambda_2 \sim \frac{c}{n^2}$, $\nu_2 \sim \frac{c}{n^2}$

walk from center. move left/right with prob $\frac{1}{2}$.

After t steps std dev is $\sim\sqrt{t}$, so expect to need

$\sqrt{t} = \frac{n}{2}$ steps, or $t = \frac{n^2}{4}$

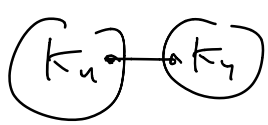
CBT: $\lambda_2, \nu_2 \sim \frac{c}{n}$

If at root, mix quickly

If at internal: go up with prob $1/3$

down with prob $2/3$

Takes about $2^d = cn$ steps to get to root

Dumbbell 

all vertices degree $n-1$ or n.

expect time $\sim n^2$ when hit bridge is $\frac{1}{n}$ chance ~~take~~ it.

prob hit bridge $\sim \frac{1}{n}$

$\lambda_2 \leq \frac{1}{2n}$: test vector (-1)—(1)

$$\lambda_2 \leq \frac{1}{2n(n-1)} = \frac{c}{n^2}$$

<u>Lem</u> Let G be connected with diameter r. Then

$$\lambda_2 \geq \frac{2}{r(n-1)}$$

<u>Proof</u> for every pair $P_{a,b}$ = path len $\leq r$.

$$L_{(a,b)} \preceq r\, L_{P(a,b)} \preceq r\, L_G$$

$$\Rightarrow K_n = \sum_{a<b} G_{a,b} \preceq \binom{n}{2} r\, L_G$$

$$\lambda_2(K_n) = n \leq \binom{n}{2} r\, \lambda_2(L_G)$$

$$\lambda_2 \geq \frac{2}{r(n-1)}$$

So, $\lambda_2(\text{Dumbell}) \geq$