Graphs and Networks	Lecture 13
Effective Resistance	
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13.1 Overview

We introduce the effective resistance between two vertices in a graph. It is a very useful measure of how close two vertices are. I becomes lower as one adds paths between vertices.

Effective resistance proves very useful in many applications. Some cool theoretical papers that use it include [Tet91, CRR$^+$96, KR93, GBS08, SS10]. Notable papers that use it in Machine Learning include [FPS05, YFD$^+$07]. Other treatments may be found in the books by Doyle and Snell [DS84] and Bollobas [Bol98].

13.2 A Quick Review

I want to quickly review the notation from last lecture. A potential/voltage on the vertices of a graph is denoted $\boldsymbol{v}$. A flow on the edges of a graph is denoted $\boldsymbol{i}$. The external flow at vertices is denoted $\boldsymbol{i}_{ext}$. The signed edge-vertex adjacency matrix is $\boldsymbol{U}$, the diagonal matix of edge weights is $\boldsymbol{W}$, and the diagonal matrix of edge resistances is $\boldsymbol{R}$. Finally,

$$\boldsymbol{L} = \boldsymbol{u}^T \boldsymbol{W} \boldsymbol{U}.$$

These satisfy the relations.

$$\begin{aligned} \boldsymbol{W} &= \boldsymbol{R}^{-1} \\ \boldsymbol{i} &= \boldsymbol{W} \boldsymbol{U} \boldsymbol{v} \\ \boldsymbol{i}_{ext} &= \boldsymbol{U}^T \boldsymbol{i} \\ \boldsymbol{i}_{ext} &= \boldsymbol{L} \boldsymbol{v}. \end{aligned}$$

At the end of last class, I mentioned that the voltages $\boldsymbol{v}$ could be found from $\boldsymbol{i}_{ext}$ by mutiplying $\boldsymbol{i}_{ext}$ by the pseudo-inverse of $\boldsymbol{L}$, denoted $\boldsymbol{L}^+$. Let me quickly say a few words about the pseudo-inverse of a symmetric matrix. For a symmetric matrix $\boldsymbol{L}$, let $\boldsymbol{\Pi}$ denote the symmetric projection matrix onto the span of $\boldsymbol{L}$. For a connected graph, this $\boldsymbol{\Pi}$ just projects a vector orthogonal to $\boldsymbol{1}$.

Definition 13.2.1. *The pseudo-inverse of $\boldsymbol{L}$, written $\boldsymbol{L}^+$, satisfies*

$$\begin{aligned} \boldsymbol{L}\boldsymbol{L}^+ &= \boldsymbol{\Pi} \\ \boldsymbol{L}\boldsymbol{L}^+\boldsymbol{L} &= \boldsymbol{L} \\ \boldsymbol{L}\boldsymbol{\Pi} &= \boldsymbol{L}. \end{aligned}$$

If there is time at the end of lecture, I'll tell you more useful facts about the pseduo-inverse. It is very useful.

We also defined the energy dissipation of an electrical $\boldsymbol{i}$ to be

$$\mathcal{E}(\boldsymbol{i}) = \boldsymbol{i}^T \boldsymbol{R} \boldsymbol{i} = \sum_e \boldsymbol{i}(e)^2 r_e = \sum_{(a,b)\in E} \frac{(\boldsymbol{v}(a) - \boldsymbol{v}(b))^2}{r_{u,v}^2} r_{u,v} = \sum_{(a,b)\in E} (\boldsymbol{v}(a) - \boldsymbol{v}(b))^2 r_{u,v} = \boldsymbol{v}^T \boldsymbol{L} \boldsymbol{v}.$$

There should really be a factor of $\frac{1}{2}$ in that definition. But, I'll drop it for convenience, and claim that this is just a change of units.

13.3 Effective Resistance

The effective resistance between vertices a and b is the resistance between a and b given by the whole network. That is, if we treat the entire network as a resistor.

To figure out what this is, recall the equation

$$\boldsymbol{i}(a,b) = \frac{\boldsymbol{v}(a) - \boldsymbol{v}(b)}{r_{a,b}},$$

which holds for one resistor. We define effective resistance through this equation. That is, we consider an electrical flow that sends one unit of current into node a and removes one unit of current from node b. We then measure the potential difference between a and b that is required to realize this current, and define this to be the effective resistance between a and b, and write it $\mathrm{R_{eff}}(a,b)$.

Algebraically, define $\boldsymbol{i}_{ext}$ to be the vector

$$\boldsymbol{i}_{ext}(c) = \begin{cases} 1 & \text{if } a = c \\ -1 & \text{if } b = c \\ 0 & \text{otherwise} \end{cases}.$$

This corresponds to a flow of 1 from a to b. We then solve for the voltages that realize this flow:

$$\boldsymbol{L}\boldsymbol{v} = \boldsymbol{i}_{ext},$$

by

$$\boldsymbol{v} = \boldsymbol{L}^+ \boldsymbol{i}_{ext}.$$

We thus have

$$\boldsymbol{v}(a) - \boldsymbol{v}(b) = \boldsymbol{i}_{ext}^T \boldsymbol{v} = \boldsymbol{i}_{ext}^T \boldsymbol{L}^+ \boldsymbol{i}_{ext}.$$

Here is another characterization of the effective resistance.

Theorem 13.3.1. *Let $\boldsymbol{i}$ be the electrical flow of one unit from vertex a to vertex b in a graph G. Then,*

$$\mathrm{R_{eff}}_{a,b} = \mathcal{E}(\boldsymbol{i}).$$

Proof. Recalling that $\boldsymbol{i}_{ext} = \boldsymbol{L}\boldsymbol{v}$, we have

$$\mathrm{R_{eff}}_{a,b} = \boldsymbol{i}_{ext}^T \boldsymbol{L}^+ \boldsymbol{i}_{ext} = \boldsymbol{v}^T \boldsymbol{L}\boldsymbol{L}^+ \boldsymbol{L}\boldsymbol{v} = \boldsymbol{v}^T \boldsymbol{L}\boldsymbol{v} = \mathcal{E}\left(\boldsymbol{i}\right).$$

□

Let's also observe that the solution is not unique. As $\mathbf{1}$ is in the nullspace of $\boldsymbol{L}$,

$$\boldsymbol{L}(\boldsymbol{v} + c\mathbf{1}) = \boldsymbol{L}\boldsymbol{v} + c\boldsymbol{L}\mathbf{1} = \boldsymbol{L}\boldsymbol{v}$$

for every c.

In the next problem set, you will prove that for every vertex c other than a and b,

$$\boldsymbol{v}(a) \geq \boldsymbol{v}(c) \geq \boldsymbol{v}(b). \tag{13.1}$$

13.4 Examples

In the case of a path graph with n vertices and edges of weight 1, the effective resistance between the extreme vertices is $n - 1$.

In general, if a path consists of edges of resistance $r(1,2), \ldots, r(n-1,n)$ then the effective resistance between the extreme vertices is

$$r(1,2) + \cdots + r(n-1,n).$$

To see this, set the potential of vertex i to

$$\boldsymbol{v}(i) = r(i,i+1) + \cdots + r(n-1,n).$$

Ohm's law then tells us that the current flow over the edge $(i, i+1)$ will be

$$\left(\boldsymbol{v}(i) - \boldsymbol{v}(i+1)\right)/r(i,i+1) = 1.$$

If we have k parallel edges between two nodes s and t of resistances $r_1, \ldots, r_k$, then the effective resistance is

$$\mathrm{R_{eff}}(s,t) = \frac{1}{1/r_1 + \cdots + 1/r_k}.$$

Again, to see this, note that the flow over the ith edge will be

$$\frac{1/r_i}{1/r_1 + \cdots + 1/r_k},$$

so the total flow will be 1.

13.5 Effective Resistance as a Distance

A distance is any function on pairs of verices such that

1. $\delta(a, a) = 0$ for every vertex a,
2. $\delta(a, b) \geq 0$ for all vertices a, b,
3. $\delta(a, b) = \delta(b, a)$, and
4. $\delta(a, c) \leq \delta(a, b) + \delta(b, c)$.

We claim that the effective resistance is a distance. The only non-trivial part to prove is the triangle inequality, (4).

Lemma 13.5.1. *Let a, b and c be vertices in a graph. Then*

$$\mathrm{R_{eff}}(a, b) + \mathrm{R_{eff}}(b, c) \geq \mathrm{R_{eff}}(a, c).$$

Proof. Let $\boldsymbol{i}_{a,b}$ be the external current corresponding to sending one unit of current from a to b, and let $\boldsymbol{i}_{b,c}$ be the external current corresponding to sending one unit of current from b to c. Note that

$$\boldsymbol{i}_{a,c} = \boldsymbol{i}_{a,b} + \boldsymbol{i}_{b,c}.$$

Now, define the corresponding voltages by

$$\boldsymbol{v}_{a,b} = \boldsymbol{L}^{+}\boldsymbol{i}_{a,b} \quad \boldsymbol{v}_{b,c} = \boldsymbol{L}^{+}\boldsymbol{i}_{b,c}. \quad \boldsymbol{v}_{a,c} = \boldsymbol{L}^{+}\boldsymbol{i}_{a,c}.$$

By linearity, we have

$$\boldsymbol{v}_{a,c} = \boldsymbol{v}_{a,b} + \boldsymbol{v}_{b,c},$$

and so

$$\mathrm{R_{eff}}(a, c) = \boldsymbol{i}_{a,c}^{T}\boldsymbol{v}_{a,c} = \boldsymbol{i}_{a,c}^{T}\boldsymbol{v}_{a,b} + \boldsymbol{i}_{a,c}^{T}\boldsymbol{v}_{b,c}.$$

By equation 13.1, we have

$$\boldsymbol{i}_{a,c}^{T}\boldsymbol{v}_{a,b} = \boldsymbol{v}_{a,b}(a) - \boldsymbol{v}_{a,b}(c) \leq \boldsymbol{v}_{a,b}(a) - \boldsymbol{v}_{a,b}(b) = \mathrm{R_{eff}}(a, b)$$

and similarly

$$\boldsymbol{i}_{a,c}^{T}\boldsymbol{v}_{b,c} \leq \mathrm{R_{eff}}(b, c).$$

The lemma follows. $\square$

13.6 Other flows

We can, of course, define non-electrical flows on graphs. A flow is just a function $\boldsymbol{j}$ such that

$$\boldsymbol{j}(c, d) = -\boldsymbol{j}(d, c).$$

A flow $\boldsymbol{j}$ is a flow of one unit from a to b if

$$\boldsymbol{U}^T \boldsymbol{j} = \boldsymbol{i}_{a,b}.$$

That is, the amount of flow entering ever vertex other than a and b equals the flow leaving, and in $\boldsymbol{j}$ one unit leaves a.

There can also be flows with no external component. These are the flows $\boldsymbol{j}$ such that

$$\boldsymbol{U}^T \boldsymbol{j} = \boldsymbol{0}.$$

These are called circulations. For example, it is easy to construct a circulation around a cycle.

Even if a flow $\boldsymbol{j}$ is not an electrical flow, we define its energy to be

$$\mathcal{E}(\boldsymbol{j}) \stackrel{\text{def}}{=} \boldsymbol{j}^T \boldsymbol{R} \boldsymbol{j} = \sum_e \boldsymbol{j}(e)^2 r_e.$$

13.7 Properties of Effective Resistance

There are two other properties of effective resistance that we would like to establish.

Theorem 13.7.1 (Thompson's Principle)**.** *Let $\boldsymbol{j}$ be any flow of value 1 from s to t. That is, let $\boldsymbol{j}$ be any solution to $\boldsymbol{i}_{s,t} = \boldsymbol{U}\boldsymbol{j}$. Then,*

$$\mathcal{E}(\boldsymbol{j}) \geq \mathrm{R_{eff}}_{s,t}.$$

So, the electrical flow is the one that minimizes the energy. Last lecture, we proved an analogous statement for potentials, in the contex of rubber bands. This one is relaed, but different.

Proof. Let $\boldsymbol{i}$ be the electrical flow of one unit from a to b, and let $\boldsymbol{v}$ the the associated potentials. Now, consider the flow

$$\boldsymbol{c} = \boldsymbol{j} - \boldsymbol{i}.$$

As both $\boldsymbol{U}^T \boldsymbol{j} = \boldsymbol{i}_{s,t}$ and $\boldsymbol{U}^T \boldsymbol{i} = \boldsymbol{i}_{s,t}$, we have

$$\boldsymbol{U}^T \boldsymbol{c} = \boldsymbol{0},$$

and so $\boldsymbol{c}$ is a circulation. We will now prove that

$$\mathcal{E}(\boldsymbol{j}) = \mathcal{E}(\boldsymbol{c}) + \mathcal{E}(\boldsymbol{i}).$$

As energy is non-negative, this implies the theorem.

We begin with the straightforward computation.

$$
\begin{aligned}
\mathcal{E}\left(\boldsymbol{j}\right) &= \sum_{(a,b)\in E} \boldsymbol{j}(a,b)^2 r_{a,b} \\
&= \sum_{(a,b)\in E} \left(\boldsymbol{i}(a,b) + \boldsymbol{c}(a,b)\right)^2 r_{a,b} \\
&= \sum_{(a,b)\in E} \left(\boldsymbol{i}(a,b)\right)^2 r_{a,b} + \sum_{(a,b)\in E} \left(\boldsymbol{c}(a,b)\right)^2 r_{a,b} + 2 \sum_{(a,b)\in E} \boldsymbol{i}(a,b)\boldsymbol{c}(a,b) r_{a,b}.
\end{aligned}
$$

The first two terms on the last line are $\mathcal{E}\left(\boldsymbol{i}\right)$ and $\mathcal{E}\left(\boldsymbol{c}\right)$, respectively. It remains to show that the last term is zero, which we do by

$$
\begin{aligned}
\sum_{(a,b)\in E} \boldsymbol{i}(a,b)\boldsymbol{c}(a,b) r_{a,b} &= \sum_{(a,b)\in E} \left(\boldsymbol{v}(a) - \boldsymbol{v}(b)\right)\boldsymbol{c}(a,b) \\
&= \sum_{(a,b)\in E} \boldsymbol{v}(a)\boldsymbol{c}(a,b) + \boldsymbol{v}(b)\boldsymbol{c}(b,a) \\
&= \sum_{a\in V} \boldsymbol{v}(a) \sum_{b:(a,b)\in E} \boldsymbol{c}(a,b) \\
&= \sum_{a\in V} 0,
\end{aligned}
$$

as $\boldsymbol{c}$ is a circulation. □

We will also prove Rayleigh's Monotonicity Principle. It seems very intuitive. However, we will see that slight extensions of it turn out to be false.

Theorem 13.7.2 (Rayleigh's Monotonicity)**.** *The effective resistance between a pair of vertices cannot be decreased by increasing the resistance of some edges.*

Proof. We first restate the theorem. Let $\boldsymbol{r}$ be a vector of resistances on edges, and let $\boldsymbol{r}'$ be another vector of resistances such that

$$\boldsymbol{r}'(a,b) \geq \boldsymbol{r}(a,b),$$

for all edges (a,b). We let $\mathcal{E}_{\boldsymbol{r}}\left(\boldsymbol{j}\right)$ denote the energy of a flow $\boldsymbol{j}$ under the resistances $\boldsymbol{r}$. For any two vertices s and t, let $\boldsymbol{i}$ be the unit electrical flow between s and t under resistances $\boldsymbol{r}$, and let $\boldsymbol{i}'$ be the same for $\boldsymbol{r}'$. The theorem says that

$$\mathcal{E}_{\boldsymbol{r}'}\left(\boldsymbol{i}'\right) \geq \mathcal{E}_{\boldsymbol{r}}\left(\boldsymbol{i}\right).$$

Let's prove it using Thompson's principle. We have

$$
\begin{aligned}
\mathcal{E}_{\boldsymbol{r}'}\left(\boldsymbol{i}'\right) &\geq \mathcal{E}_{\boldsymbol{r}}\left(\boldsymbol{i}'\right) && \text{as } \boldsymbol{r}' \geq \boldsymbol{r} \\
&\geq \mathcal{E}_{\boldsymbol{r}}\left(\boldsymbol{i}\right) && \text{by Thompson's principle.}
\end{aligned}
$$

□

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