Graphs and Networks	Lecture 8
Resistor Networks and Random Walks	
Lecturer: Daniel A. Spielman	October 2, 2007

8.1 Introduction

In this lecture and the next, we are going to consider a physical model of a graph in which each edge is a resistor. We will then consider attaching vertices to terminals of a battery, and examine the current that flows. We will see that there is a relation between the induced voltages, and random walks in a graph. We will also see how to compute the induced voltages by solving systems of equations in Laplacians.

8.2 Resistor Networks

Given a graph, we can treat each edge as a resistor. If the graph is unweighted, we will assume that the resistor has resistance 1. If an edge e has weight $w(e)$, we will give the corresponding resistor resistance $r(e) = 1/w(e)$. The reason is that when the weight of an edge is very small, the edge is barely there, so it should correspond to very high resistance. Havin no edge corresponds to having a resistor of infinite resistance.

Now, we'll reach deep back into our memories to recall the laws governing electrical currents in resistor networks. Assume that we are going to inject and withdraw current from the network. We let $\boldsymbol{i}_{ext}(x)$ be the amount of current that we are going to inject into node x. If this quantity is negative then we are removing current. As a resistor network cannot hold current, we must inject as much as we remove:

$$\sum_x \boldsymbol{i}_{ext}(x) = 0. \tag{8.1}$$

When we inject current into the network, a current is naturally induced in each edge. For an edge (x, y), we let $\boldsymbol{i}(x, y)$ denote the current flowing from x to y. As this is a directed quantity, we have

$$\boldsymbol{i}(x, y) = -\boldsymbol{i}(y, x).$$

Kirchoff's Current Law tells us that no current flows in or out of any vertex, other than specified by $\boldsymbol{i}_{ext}$:

$$\forall x \in V, \qquad \sum_{y:(x,y)\in E} \boldsymbol{i}(x, y) = \boldsymbol{i}_{ext}(x).$$

In particular, we get zero above if we are not injecting any current into vertex x.

Recall that the induced electrial flow is a *potential flow*. That means that the flow is determined by a vector of potentials $\boldsymbol{v} \in \mathbb{R}^V$. Ohm's Law provides a relation between the difference in potentials across an edge and the amount of current that flows across it:

$$\boldsymbol{i}(x,y) = (\boldsymbol{v}(x) - \boldsymbol{v}(y))/r(x,y).$$

By combining Ohm's Law with Kirchoff's Current Law, we get the following relationship between the injected currents and the induced potentials:

$$\begin{aligned}\boldsymbol{i}_{ext}(x) &= \sum_{y:(x,y)\in E} \boldsymbol{i}(x,y)\\ &= \sum_{y:(x,y)\in E} (\boldsymbol{v}(x) - \boldsymbol{v}(y))/r(x,y)\\ &= \sum_{y:(x,y)\in E} (\boldsymbol{v}(x) - \boldsymbol{v}(y))\, w(x,y),\end{aligned}$$

where $w(x,y)$ is the weight of edge (x,y) in the graph. We can simplify this expression a little more. Recall the definion of the weighted degree of a vertex of a graph:

$$d(x) = \sum_{y:(x,y)\in E} w(x,y).$$

Using this definition, we can rewrite the above equation as

$$\boldsymbol{i}_{ext}(x) = d(x)\boldsymbol{v}(x) - \sum_{y:(x,y)\in E} \boldsymbol{v}(y)w(x,y). \tag{8.2}$$

That is,

$$\boldsymbol{i}_{ext} = \boldsymbol{L}\boldsymbol{v}, \tag{8.3}$$

where $\boldsymbol{L}$ is the Laplacian matrix of the weighted graph.

8.3 Examples

I'll do examples on the path graph, two parallel paths, and two parallel paths with a tail.

8.4 Solving for voltages

We now observe that we can always solve equation (8.3) for $\boldsymbol{v}$, provided that (8.1) holds and the underlying graph is connected. Typically, equations like (8.3) are solvable, except when the matrix is degenerate. Unfortunately, $\boldsymbol{L}$ does have determinant zero, and so is not necessarily invertible.

But, we know a lot about $\boldsymbol{L}$: its nullspace is spanned by $\mathbf{1}$. Equation (8.1) tells us that $\boldsymbol{i}_{ext}$ is orthogonal to the nullspace of $\boldsymbol{L}$, and so, because $\boldsymbol{L}$ is symmetric, $\boldsymbol{i}_{ext}$ is in the range of $\boldsymbol{L}$. So, we can solve (8.3) by just inverting $\boldsymbol{L}$ on its range–which is the space orthogonal to its nullspace. This will provide us with a solution $\boldsymbol{v}$ that is also orthogonal to the nullspace.

Here's a more explicit way of constructing the solution. Let $0 = \gamma_1 < \gamma_2 \leq \cdots \leq \gamma_n$ be the eigenvalues of $\boldsymbol{L}$ and let $\mathbf{1} = \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ be a corresponding orthonormal basis of eigenvectors. We have

$$\boldsymbol{i}_{ext} = \sum_{i=1}^{n} \boldsymbol{u}_i \left(\boldsymbol{u}_i^T \boldsymbol{i}_{ext}\right).$$

As

$$\left(\boldsymbol{u}_i^T \boldsymbol{i}_{ext}\right) = 0,$$

we can simplify this to

$$\boldsymbol{i}_{ext} = \sum_{i=2}^{n} \boldsymbol{u}_i \left(\boldsymbol{u}_i^T \boldsymbol{i}_{ext}\right).$$

So, we can just set

$$\boldsymbol{v} = \sum_{i=2}^{n} \boldsymbol{u}_i \left(\boldsymbol{u}_i^T \boldsymbol{i}_{ext}\right) / \gamma_i.$$

This will be a solution to (8.3) because

$$\begin{aligned}
\boldsymbol{L}\boldsymbol{v} &= \boldsymbol{L} \sum_{i=2}^{n} \boldsymbol{u}_i \left(\boldsymbol{u}_i^T \boldsymbol{i}_{ext}\right) / \gamma_i \\
&= \sum_{i=2}^{n} \gamma_i \boldsymbol{u}_i \left(\boldsymbol{u}_i^T \boldsymbol{i}_{ext}\right) / \gamma_i \\
&= \sum_{i=2}^{n} \boldsymbol{u}_i \left(\boldsymbol{u}_i^T \boldsymbol{i}_{ext}\right) \\
&= \boldsymbol{i}_{ext}.
\end{aligned}$$

The bottom line is that the conventional wisdom that "one can only solve a system of linear equations if the matrix is invertible" is just too pesimistic. The more optimistic approach that we have taken here is to solve the system by multiplying by the pseudo-inverse instead of the inverse. Since you may find it useful later in life, let me say that the pseudo-inverse of a symmetric matrix is just the inverse on the range of the matrix. For a matrix with eigenvalues $\gamma_1, \ldots, \gamma_n$ and corresponding eigenvectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$, it is given by

$$\boldsymbol{L}^{+} \stackrel{\text{def}}{=} \sum_{i:\gamma_i \neq 0} \gamma_i \boldsymbol{u}_i \boldsymbol{u}_i^T.$$

8.5 Fixing Potentials

We usually don't think of injecting current into a circuit. Rather, we usually attach nodes of a circuit to the terminals of a battery, which induce fixed potentials. Typically, we just attach two vertices to the terminals. Let's call them s and t, and assume that s has been attached to a terminal of potential 1 and that t has been attached to a terminal of potential 0. That is, $\boldsymbol{v}(s) = 1$ and

$\boldsymbol{v}(t) = 0$. Then, the potentials all the remaining vertices can be determined by obvserving that for $x \notin \{s,t\}$, $\boldsymbol{i}_{ext}(x) = 0$ and applying (8.2) to show

$$d(x)\boldsymbol{v}(x) = \sum_{y:(x,y)\in E} \boldsymbol{v}(y)w(x,y). \tag{8.4}$$

Thus, we get another system of linear equations, with boundary conditions fixing the individual variables at s and t. We could also use linear algebra to prove that these equations have a solution. I'll give you a break and use probability instead. We will first show that the equations have a solution, and then prove that it is unique.

8.6 Random Walks

Consider a random walk that starts at a vertex x, and stops when it finally reaches s or t. Let $F(x)$ be the probability that it hits s before t. It should be clear that F is well-defined, and that by definition $F(s) = 1$ and $F(t) = 0$.

Let's see what happens in between. For a vertex x that is neither s nor t, the probability that it steps to a node y is given by

$$w(x,y)/d(x).$$

We have

$$\begin{aligned}
F(x) &= \sum_{y:(x,y)\in E} \Pr\left[\text{the walk goes to } y \text{ after } x\text{, and eventually stops at } s\right] \\
&= \sum_{y:(x,y)\in E} \Pr\left[\text{the walk goes to } y \text{ after } x\right] F(y) \\
&= \sum_{y:(x,y)\in E} F(y)w(x,y)/d(x).
\end{aligned}$$

Multiplying through by $d(x)$, we get

$$d(x)F(x) = \sum_{y:(x,y)\in E} F(y)w(x,y).$$

That is, F is a solution to equation (8.4).

Let me point out that one can generalize this technique to find a solution when any number of vertices are fixed to any potentials. Let $W \subset V$ be the set of vertices whose potential we will fix. Consider fixing the potential of any element of W to 1 and the rest to 0. We can then enforce (8.4) at the rest of the vertices by considering the random walk that stops when it reaches any node of W, and considering the probability that it hits s first. Once you know how to build this function, you can build any function on W by taking a linear combination of such functions.

8.7 Uniqueness

Let's now prove that the solutions to these equations are unique if the graph is connected. Given $W \subset V$, we say that a function $f : V \to \mathbb{R}$ is *harmonic* on $V - W$ if for all $x \in V - W$,

$$d(x)f(x) = \sum_{y:(x,y)\in E} f(y)w(x,y).$$

We will now show that a harmonic function is completely determined by its values on W.

Lemma 8.7.1. *Let f and g be two functions that are both harmonic on $V - W$ and such that for all $w \in W$*

$$f(w) = g(w).$$

Then, for all $x \in V - W$,

$$f(x) = g(x).$$

Proof. Let $h = f - g$. So, $h(w) = 0$ for all $w \in W$, and we need to show that $h(x) = 0$ for all $x \in V$.

Let z be such that $h(x) \leq h(z)$ for all $x \in V$. We have

$$h(z) = \sum_{y:(z,y)\in E} h(y)w(z,y)/d(z).$$

As $\sum_y w(z,y)/d(z) = 1$ and $w(z,y) > 0$, the right-hand side is a weighted average of the values $h(y)$. As it is also equal to the maximum of h, all the values $h(y)$ must be equal. So, $h(y) = h(z)$ for every neighbor y of z. By induction on paths in the graph, $h(x) = h(z)$ for every vertex x reachable from z. As this holds for some vertex $w \in W$, we have $h(z) = h(w) = 0$, and $h(x) = 0$ for all vertices x that are reachable from z, which is all of V provided that the graph is connected. $\square$

8.8 Segmentation by Electrical Flows

Grady (IEEE T. PAMI 2006) suggests that one can use electrical flows to partition a graph. Say that one knows two vertices, s and t that one would like to put on different sides of the partition. Then, one can induce a function on the graph by setting $\boldsymbol{v}(s) = 1$, $\boldsymbol{v}(t) = 0$, and letting the remaining vertices have the induced potential. One can then look for a cut among the sets of the form

$$\{x : \boldsymbol{v}(x) \leq t\},$$

for some threshold t.

Of course, one can also fix many vertices to fixed potentials. For example, if one knows that one set of vertices S should go together and that another set of vertices T should go together, then one can fix

$$\boldsymbol{v}(x) = \begin{cases} 1 & \text{for } x \in S \\ 0 & \text{for } x \in T, \end{cases}$$

and set $\boldsymbol{v}(x)$ by (8.4) otherwise.

Note that that is also equivalent to letting each edge be a rubber band, fixing all the vertices in s and t, and letting the rest settle.