Effective Resistance and Random Walks

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October 9, 2007

10.1 Flows

We think of a flow as a function that tells us how much current passes through each edge, and in which direction. The assignment of direction is a little awkward because we have been treating the edges as being undirected. In this lecture, we will need to be more careful. So, we will replace each undirected edge by two directed edges, and call the resulting set of edges $\overrightarrow{\mathrm{E}}$.

We then define a *flow* to be any function $\boldsymbol{i} : \overrightarrow{\mathrm{E}} \to \mathbb{R}$ for which $\boldsymbol{i}(x, y) = -\boldsymbol{i}(y, x)$. We say that a flow is an *st-flow* if no flow leaves the circuit at any vertex other than s or t. That is, if

$$\forall u \notin \{s, t\}, \qquad \sum_{v:(u,v)\in \overrightarrow{\mathrm{E}}} \boldsymbol{i}(u, v) = 0. \tag{10.1}$$

10.2 Energy

Given a flow $\boldsymbol{i}$ on the edges of a graph, we define the *energy* (or energy dissipation) of the flow to be

$$\mathcal{E}(\boldsymbol{i}) = (1/2) \sum_{e\in \overrightarrow{\mathrm{E}}} \boldsymbol{i}(e)^2 r(e).$$

The reason for the one-half in the above expression is that we are counting each edge twice, as the sum equals

$$\sum_{e\in E} \boldsymbol{i}(e)^2 r(e).$$

In the next section, we will prove that the unit electrical flow between s and t is the flow of minimum energy. First, we will prove that the energy of the unit electrical flow equals the effective resistance between s and t. (You might want to check this for the path graph).

The proof will use the following striking but trivial lemma.

Lemma 10.2.1. *Let $f \in \mathbb{R}^V$ be any function on the vertices of a graph and let $\boldsymbol{i}$ be any st-flow on its edges. Set*

$$\boldsymbol{i}_{ext}(u) = \sum_{v:(u,v)\in \overrightarrow{\mathrm{E}}} \boldsymbol{i}(u, v),$$

so $\boldsymbol{i}_{ext}(u) = 0$ *for all* $u \notin \{s, t\}$. *Then*

$$\boldsymbol{i}_{ext}(s)(f(s) - f(t)) = (1/2) \sum_{(u,v) \in \overrightarrow{\mathrm{E}}} \boldsymbol{i}(u, v)(f(u) - f(v)). \tag{10.2}$$

Before proving this lemma, let's see what it says about electrical flows.

Corollary 10.2.2. *Let* $\boldsymbol{i}$ *be the unit electrical flow between two vertices* s *and* t *in a graph. Then,*

$$\mathrm{R_{eff}}(s, t) = \mathcal{E}(\boldsymbol{i}).$$

Proof. Let $\boldsymbol{v}$ be the potential realizing the electrical flow in which $\boldsymbol{v}(s) = \mathrm{R_{eff}}(s, t)$ and $\boldsymbol{v}(t) = 0$. By Ohm's law, the current flow over each edge (x, y) satisfies

$$r(x, y)\boldsymbol{i}(x, y) = (\boldsymbol{v}(x) - \boldsymbol{v}(y)).$$

So, plugging $\boldsymbol{v}$ and $\boldsymbol{i}$ into Lemma 10.2.1, we get

$$\begin{aligned}\mathrm{R_{eff}}(s, t) = \boldsymbol{i}_{ext}(s)(\boldsymbol{v}(s) - \boldsymbol{v}(t)) = (1/2) \sum_{(x,y) \in \overrightarrow{\mathrm{E}}} \boldsymbol{i}(x, y)(\boldsymbol{v}(x) - \boldsymbol{v}(y)) \\ = (1/2) \sum_{(x,y) \in E} \boldsymbol{i}(x, y)^2 r(x, y) = \mathcal{E}(\boldsymbol{i}).\end{aligned}$$

□

Proof of Lemma 10.2.1. We have

$$\begin{aligned}\sum_{(u,v) \in \overrightarrow{\mathrm{E}}} \boldsymbol{i}(u, v)\left(f(u) - f(v)\right) &= \sum_{u \in V} \sum_{v:(u,v) \in \overrightarrow{\mathrm{E}}} \boldsymbol{i}(u, v) f(u) - \sum_{v \in V} \sum_{u:(u,v) \in \overrightarrow{\mathrm{E}}} \boldsymbol{i}(u, v) f(v) \\ &= 2 \sum_{u \in V} \sum_{v:(u,v) \in \overrightarrow{\mathrm{E}}} \boldsymbol{i}(u, v) f(u) \\ &= 2\left[\boldsymbol{i}_{ext}(s) f(s) - \boldsymbol{i}_{ext}(t) f(t)\right] \\ &= 2\boldsymbol{i}_{ext}(s)\left(f(s) - f(t)\right),\end{aligned}$$

where the second equality follows by $\boldsymbol{i}(u, v) = -\boldsymbol{i}(v, u)$, the third follows from $\boldsymbol{i}_{ext}(u) = 0$ for $u \notin \{s, t\}$, and the last follows from $\boldsymbol{i}_{ext}(s) = -\boldsymbol{i}_{ext}(t)$. □

10.3 Thompson's Principle

Thompson's principle says that the minimum energy unit st-flow is the electrical flow. Let's prove it.

Theorem 10.3.1. *Let* $\boldsymbol{j}$ *be any st-flow. Then,*

$$\mathcal{E}(\boldsymbol{j}) \geq \mathrm{R_{eff}}(s, t).$$

Proof. Let $\boldsymbol{i}$ be the unit st-electrical flow, and let $\boldsymbol{v}$ be the associated potential. Let $\boldsymbol{c} = \boldsymbol{j} - \boldsymbol{i}$. Then, $\boldsymbol{c}$ is a flow with no external flow. That is, $\boldsymbol{c}$ is a circulation. We have

$$\begin{aligned}
\mathcal{E}(\boldsymbol{j}) &= \sum_{(x,y)\in E} \boldsymbol{j}(x,y)^2 r(x,y) \\
&= \sum_{(x,y)\in E} (\boldsymbol{i}+\boldsymbol{c})(x,y)^2 r(x,y) \\
&= \sum_{(x,y)\in E} \boldsymbol{i}(x,y)^2 r(x,y) + 2\sum_{(x,y)\in E} \boldsymbol{i}(x,y)\boldsymbol{c}(x,y) r(x,y) + \sum_{(x,y)\in E} \boldsymbol{c}(x,y)^2 r(x,y) \\
&= \mathcal{E}(\boldsymbol{i}) + 2\sum_{(x,y)\in E} \boldsymbol{c}(x,y)(\boldsymbol{v}(x)-\boldsymbol{v}(y)) + \mathcal{E}(\boldsymbol{c}).
\end{aligned}$$

Now, by Lemma 10.2.1,

$$2\sum_{(x,y)\in E} \boldsymbol{c}(x,y)(\boldsymbol{v}(x)-\boldsymbol{v}(y)) = 0,$$

as no flow leaves or enters s in $\boldsymbol{c}$. So, we get

$$\mathcal{E}(\boldsymbol{j}) = \mathcal{E}(\boldsymbol{i}) + \mathcal{E}(\boldsymbol{c}).$$

□

10.4 Rayleigh's Monotonicity Principle

Rayleight's monotonicity principle says that...

Theorem 10.4.1. *If the resistance of any edge in a network is increased, then for every s and t, the effective resistance between s and t does not decrease.*

Proof. Let $\mathcal{E}'$ denote the energy in the modified network, and let R'_{eff} denote the effective resistance between s and t in the modified network. For every st-flow $\boldsymbol{i}$,

$$\mathcal{E}'(\boldsymbol{i}) \geq \mathcal{E}(\boldsymbol{i}).$$

Let $\boldsymbol{i}$ be the unit electrical st-flow in the modified network, so that

$$R'_{eff}(s,t) = \mathcal{E}'(\boldsymbol{i}) \geq \mathcal{E}(\boldsymbol{i}) \geq \mathrm{R}_{\mathrm{eff}}(s,t),$$

where the last inequality follows from Lemma 10.3.1. □

10.5 Commute times and Hitting times

Let $t \in V$ and view t as the *target* of a walk in the graph G. For every node $v \in V$, we will let $H_t(v)$ denote expected number of steps it takes a random walk from v to hit t. This is often called the *hitting time*. The commute time between s and t is the expected number of steps it takes a random walk that starts at s to hit t and then return to s. It is denoted $C(s,t)$, and clearly satisfies

$$C(s,t) = H_t(s) + H_s(t).$$

Note that $H_t(t) = 0$, and for $x \neq t$, we have

$$H_t(x) = \sum_{y:(x,y)\in \overrightarrow{\mathrm{E}}} \frac{w(x,y)}{d(x)} (1 + H_t(y)). \tag{10.3}$$

That is, the expected time to hit t from x is the average over the neighbors of x of one plus the expected time to hit t from that neighbor.

We will now show:

Theorem 10.5.1.

$$C(s,t) = d(V)\mathrm{R_{eff}}(s,t).$$

Recall that $d(V)$ is the sum of the weighted degrees of vertices in V. The proof will follow an examination of the flow when a carefully chosen external current is injected.

First, fix the vertex t. Set

$$\boldsymbol{i}^T_{ext}(x) = \begin{cases} d(x) & \text{if } x \neq t \\ d(t) - d(V) & \text{if } x = t. \end{cases}$$

Note that

$$\mathbf{1}\boldsymbol{i}_{ext} = 0.$$

Let $\boldsymbol{v}$ be the solution to the equation

$$\boldsymbol{L}\boldsymbol{v} = \boldsymbol{i}_{ext}$$

in which $\boldsymbol{v}(t) = 0$. (Recall that for every solution $\boldsymbol{v}$, $\boldsymbol{v} + c\mathbf{1}$ is also a solution.)

I will now show you that for every vertex x

$$\boldsymbol{v}(x) = H_t(x).$$

First, note that it trivially holds at $x = t$. Next, recall that $\boldsymbol{v}$ is the unique solution to the equations

$$\begin{aligned} d(x) &= \boldsymbol{i}_{ext}(x) \\ &= \sum_{(x,y)\in \overrightarrow{\mathrm{E}}} \boldsymbol{i}(x,y) \\ &= \sum_{(x,y)\in \overrightarrow{\mathrm{E}}} (\boldsymbol{v}(x) - \boldsymbol{v}(y))/r(x,y) \\ &= \sum_{(x,y)\in \overrightarrow{\mathrm{E}}} (\boldsymbol{v}(x) - \boldsymbol{v}(y))w(x,y), \end{aligned}$$

for which $\boldsymbol{v}(t) = 0$.

To show that H_t also satisfies this equation, and this equals $\boldsymbol{v}$, we multiply both sides of equation (10.3) by $d(x)$ to get

$$\begin{aligned} d(x)H_t(x) &= \sum_{y:(x,y)\in\overrightarrow{\mathrm{E}}} w(x,y)(1 + H_t(y)) \\ &= d(x) + \sum_{y:(x,y)\in\overrightarrow{\mathrm{E}}} w(x,y)H_t(y), \end{aligned}$$

which implies

$$d(x) = d(x)H_t(x) - \sum_{y:(x,y)\in\overrightarrow{\mathrm{E}}} w(x,y)H_t(y) = \sum_{y:(x,y)\in\overrightarrow{\mathrm{E}}} w(x,y)(H_t(x) - H_t(y)).$$

Now, let's finish the proof of the theorem.

Define $\boldsymbol{j}_{ext}$ to be the flow in which we inject $d(V) - d(s)$ units of current into s, and remove $d(x)$ units from every other vertex x. Note that this is the opposite of how we defined $\boldsymbol{i}_{ext}$. Let $\boldsymbol{w}$ be the solution to

$$\boldsymbol{L}\boldsymbol{w} = \boldsymbol{j}_{ext}$$

in which $\boldsymbol{w}(s) = 0$. We then have that $\boldsymbol{w}(x) < 0$ for all other x, and

$$H_s(x) = -\boldsymbol{w}(x).$$

Now, consider the external flow

$$\boldsymbol{k}_{ext} = \boldsymbol{i}_{ext} + \boldsymbol{j}_{ext}.$$

We have arranged that $\boldsymbol{k}_{ext}$ is an st-flow of $d(V)$ units of current. Moreover, a solution to

$$\boldsymbol{L}\boldsymbol{z} = \boldsymbol{k}_{ext}$$

is given by

$$\boldsymbol{z} = \boldsymbol{v} + \boldsymbol{w},$$

and

$$\boldsymbol{z}(s) = H_t(s) \qquad \text{and} \qquad \boldsymbol{z}(t) = -H_s(t).$$

Recall that $\mathrm{R_{eff}}(s,t)$ is the difference in potential between s and t in the st-flow that sends 1 unit of current. So,

$$d(V)\mathrm{R_{eff}}(s,t) = (\boldsymbol{z}(s) - \boldsymbol{z}(t)) = H_t(s) + H_s(t).$$

10.6 Mean Return Time

Let me finish this lecture with a surprising fact we can obtain from the analysis technique we just used. Let $H(x)$ be the average amount of time it takes a random walk from vertex x to return to vertex x, assuming that it must leave. That is, we don't take the trivial solution of 0. We will now prove

$$H(s) = \frac{d(V)}{d(s)},$$

for all vertices s. That is, the mean return time barely depends upon the graph!

To prove this, first observe that

$$H(s) = \sum_{y:(s,y)\in\overrightarrow{\mathrm{E}}} (1 + H_s(y)) \frac{w(s,y)}{d(s)}.$$

Let $\boldsymbol{j}_{ext}$ and $\boldsymbol{w}$ be as in the previous section. So, we have

$$\begin{aligned}
d(V) &= d(s) + \boldsymbol{i}_{ext}(s) \\
&= d(s) + \sum_{x:(s,x)\in\overrightarrow{\mathrm{E}}} \boldsymbol{i}(s,x) \\
&= d(s) + \sum_{x:(s,x)\in\overrightarrow{\mathrm{E}}} (\boldsymbol{w}(s) - \boldsymbol{w}(x)) w(s,x) \\
&= d(s) + \sum_{x:(s,x)\in\overrightarrow{\mathrm{E}}} H_s(x) w(s,x) \\
&= \sum_{x:(s,x)\in\overrightarrow{\mathrm{E}}} (1 + H_s(x)) w(s,x) \\
&= d(s) \sum_{x:(s,x)\in\overrightarrow{\mathrm{E}}} (1 + H_s(x)) \frac{w(s,x)}{d(s)}. \\
&= d(s) H(s).
\end{aligned}$$