Simultaneous Games

payoff for I

		Player II		
		Rock	Paper	Scissors
Player I	Rock	0	-1	1
	Paper	1	0	-1
	Scissors	-1	1	0

P(I plays Rock) P(I plays paper) P(I plays scissors)

$$\frac{4}{10} \quad \frac{3}{10} \quad \frac{3}{10}$$

Expected payoff for I when II plays

$$\text{Rock} = \frac{4}{10}\cdot 0 + \frac{3}{10}\cdot 1 + \frac{3}{10}\cdot -1 = 0$$

$$\text{Paper} = \frac{4}{10}\cdot -1 + \frac{3}{10}\cdot 0 + \frac{3}{10}\cdot 1 = -\frac{1}{10}$$

$$\text{Scissors} = \frac{4}{10}\cdot 1 + \frac{3}{10}\cdot -1 + \frac{3}{10}\cdot 0 = \frac{1}{10}$$

dot product

$$\begin{pmatrix} \frac{4}{10} & \frac{3}{10} & \frac{3}{10} \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{10} & \frac{1}{10} \end{pmatrix}$$

vector (1-row matrix) of probabilities

payoff matrix

payoffs for each option for II

$$\underbrace{\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}}_{1 \times n} \underbrace{\begin{pmatrix} b_{11} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2p} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{np} \end{pmatrix}}_{n \times p} = \underbrace{\begin{pmatrix} c_1 & c_2 & \cdots & c_p \end{pmatrix}}_{1 \times p}$$

R P

1×3 $\quad 3 \times 3$ $\quad 1 \times 3$

where $c_j = \sum_{k=1}^{n} a_k \cdot b_{kj}$

m-row, n-col $\quad$ n-row, p-col $\quad$ m-row, p-col

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & \cdots & & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2p} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1p} \\ \vdots & & \vdots \\ c_{m1} & \cdots & c_{mp} \end{pmatrix}$$

where $c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$

dot product of row i of 1st matrix col j of 2nd matrix

$$\underbrace{\begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 0 \end{pmatrix}}_{2 \times 3} \cdot \underbrace{\begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 2 & 1 \end{pmatrix}}_{3 \times 2} = \begin{pmatrix} 8 & 3 \\ 3 & 1 \end{pmatrix}$$

$2\cdot 0 + -1\cdot -1 + 0\cdot 1$

$2\cdot 2 + -1\cdot 1 + 0\cdot 2$

$2\cdot 2 + -1\cdot 1 + 0\cdot 2$

$$\begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

$3\times 2 \quad 2\times 3 \quad 3\times 3$

$$\begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 3 \\ 2 & -1 & 0 \\ 1 & -1 & 4 \end{pmatrix}$$

$3\times 2 \quad 4\times 3$

$3\times 2 \cdot 2\times 4$

$2\times 4 \cdot 3\times 2$

$2\cdot 0 + 1\cdot 2$

$$\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & \\ & \end{pmatrix}$$

Matrix multiplication is not commutative — not the case that $A\cdot B = B\cdot A$ always

I's prob. dist. over actions

II's prob dist

$$\begin{pmatrix} \frac{4}{10} & \frac{3}{10} & \frac{3}{10} \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{10} \\ \frac{4}{10} \\ \frac{3}{10} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{10} & \frac{1}{10} \end{pmatrix} \begin{pmatrix} \frac{3}{10} \\ \frac{4}{10} \\ \frac{3}{10} \end{pmatrix}$$

$$= -\frac{1}{100}$$

$$\begin{pmatrix} \frac{4}{10} & \frac{3}{10} & \frac{3}{10} \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{10} \\ 0 \\ \frac{1}{10} \end{pmatrix} = -\frac{1}{100}$$

Matrix multiplication is associative

Inverse Matrix

$$\overset{A^{-1}}{\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{7}{3} & \frac{1}{3} & -1 \\ \frac{-5}{3} & -\frac{2}{3} & 1 \end{pmatrix}} \overset{A}{\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 3 & 1 & 2 \end{pmatrix}} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 \\ 7 & 1 & -3 \\ -5 & -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 3 & 1 & 2 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$-5\cdot 1 + -2\cdot 2 + 3\cdot 3 = 0$

$-5\cdot 1 + -2\cdot -1 + 3\cdot 1$

$-5\cdot 1 + -2\cdot -1 + 3\cdot 2$

I_3 3x3 identity matrix

$A\cdot I_n = A$

$I_n\cdot A = A$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & \vdots & & \\ a_{31} & \cdots & & a_{34} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & & & \\ a_{31} & \cdots & & a_{34} \end{pmatrix}$$

for square $n\times n$ matrix A, the inverse A^{-1} is matrix s.t. $A^{-1}\cdot A = I_n$

$A\cdot A^{-1} = I_n$

$$x_1 + x_2 + x_3 = 3$$
$$2x_1 - x_2 - x_3 = 0$$
$$3x_1 + x_2 + 2x_3 = 9$$

$x_1 = 1$
$x_2 = -2$
$x_3 = 4$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + x_3 \\ 2x_1 - x_2 - x_3 \\ 3x_1 + x_2 + 2x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 9 \end{pmatrix}$$

Find x_1, x_2, x_3 s.t.

$$\overset{A}{\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 3 & 1 & 2 \end{pmatrix}} \overset{\vec{x}}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \begin{pmatrix} 3 \\ 0 \\ 9 \end{pmatrix}$$

$4x = 9$

$$A\cdot\vec{x} = \begin{pmatrix} 3 \\ 0 \\ 9 \end{pmatrix}$$

$$A^{-1}\cdot A\cdot\vec{x} = A^{-1}\begin{pmatrix} 3 \\ 0 \\ 9 \end{pmatrix}$$

$$I_3\cdot\vec{x} = A^{-1}\begin{pmatrix} 3 \\ 0 \\ 9 \end{pmatrix}$$

$$\vec{x} = A^{-1}\begin{pmatrix} 3 \\ 0 \\ 9 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{7}{3} & \frac{1}{3} & -1 \\ \frac{-5}{3} & -\frac{2}{3} & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

$x_1 = 1$
$x_2 = -2$
$x_3 = 4$

$x_2 = -2$
$x_3 = 4$

Solve
$$\begin{cases} x_1 + x_2 + x_3 = 4 \\ x_1 + x_2 + 2x_3 = 8 \\ -3x_3 = 12 \end{cases}$$

$x_1 + x_2 = 0$
$x_1 + x_2 = 0$
$x_3 = 4$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 9 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B \cdot \overset{A}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}} = I_3$$

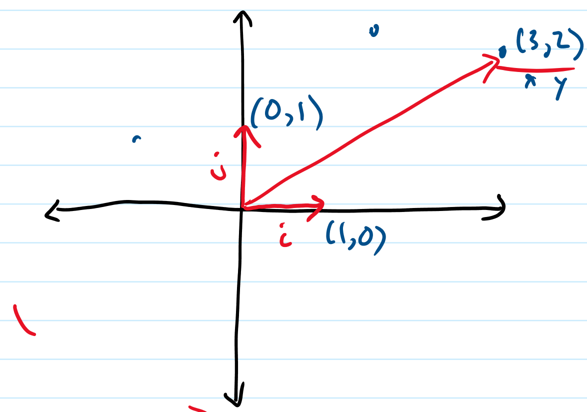
not invertible

row 3 = 3·(row 2 − row 1)
0 0 1

one of the rows is a linear combo of other rows (not linearly independent)

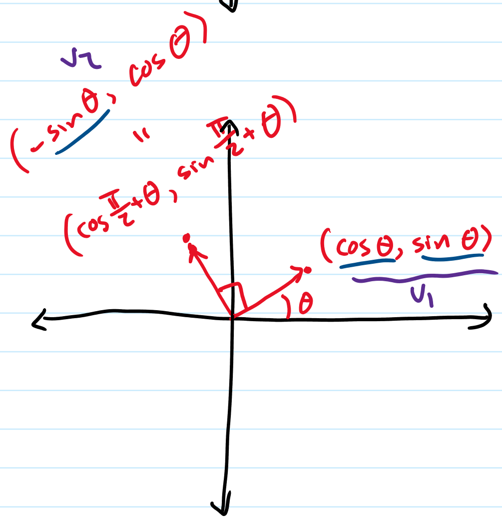
A square matrix is invertible if and only if its rows (and columns) are linearly independent

Affine Transformations



$(3,2) = 3\cdot\vec{i} + 2\cdot\vec{j}$

basis: set of linearly independent vectors



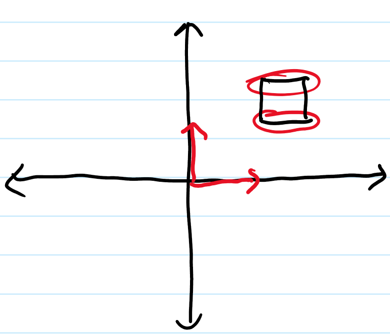
$$x\cdot\vec{v}_1 + y\cdot\vec{v}_2 = \begin{pmatrix} x\cdot\cos\theta - y\sin\theta, \\ x\cdot\sin\theta + y\cos\theta \end{pmatrix}$$

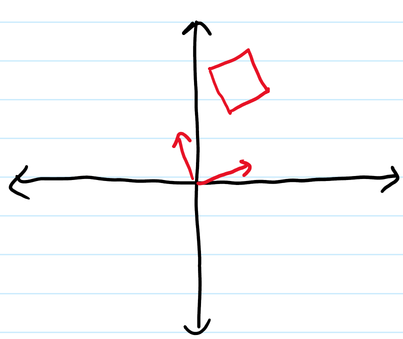
$$\begin{array}{cc} v_1 & v_2 \\ \downarrow & \downarrow \end{array}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

basis: set of vectors s.t. every vector can be represented as a

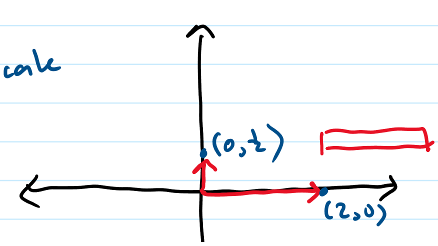
Linear Transformations

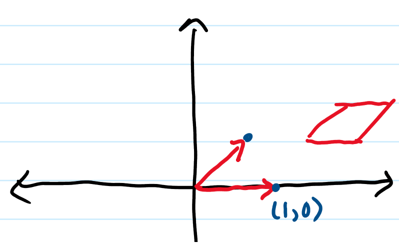




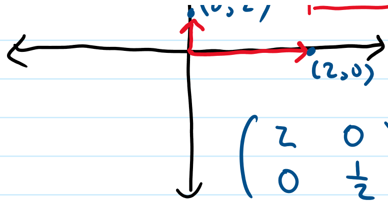
rotation

scale

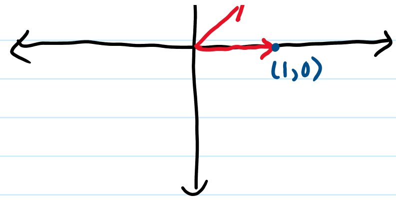




shear



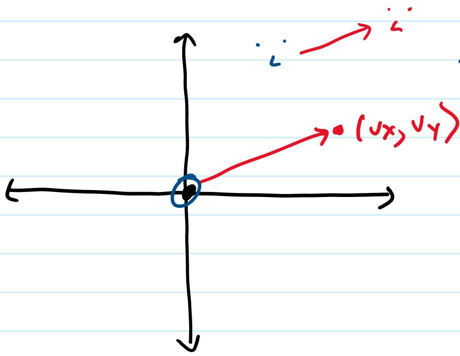
$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$



$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

every linear transformation maps the origin to origin



translation

Affine Transformation

translation

$$\begin{pmatrix} 1 & 0 & v_x \\ 0 & 1 & v_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + v_x \\ y + v_y \\ 1 \end{pmatrix}$$

rotation

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

scale $\begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} s_x \cdot x \\ s_y \cdot y \\ 1 \end{pmatrix}$

shear $\begin{pmatrix} 1 & m_x & 0 \\ m_y & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$

$T_{-1,-1}$ translate $-1, -1$

R_{60} rotate 60

$T_{1,1}$ translate $1, 1$

$T_{1,1} \left(R_{60} \left(T_{1,-1} \ \vec{x} \right) \right)$

$\left(T_{1,1} \cdot R_{60} \cdot T_{1,-1} \right) \vec{x}$