Crazy Function

Define $f: \mathbb{Z}^+ \to \mathbb{R}^+$ by $f(n) = \begin{cases} \cancel{2^n} & \cancel{\text{if } n \le 100} \\ 2^{100} \cdot n^4 & \text{otherwise} \end{cases}$

Is $f(n) \in O(2^n)$ Y $\quad \forall n \ge \underbrace{101}_{n_0},\ 2^{100} \cdot n^4 \le \underbrace{2^{100}}_{c} \cdot 2^n$ b/c for $n \ge 101$ $n^4 \le 2^n$

$f(n) \in \Theta(2^n)$ N $\quad \exists n_0, c$ s.t. $\forall n \ge n_0$ $f(n) \le c \cdot 2^n$

$f(n) \in O(2^n)$

$f(n) \in O(n^4)$ Y

$f(n) \in \Theta(n^4)$ Y $\quad \forall n \ge 101,\ \underline{1} \cdot n^4 \le \overbrace{2^{100} \cdot n^4}^{f(n)} \le \underline{2^{100}} \cdot n^4$

Properties

$2^{100} n^4 \in \Theta(n^4)$ $\qquad$ $n^4 \in \Theta(2^{60} \cdot n^4)$

THM: For any function $f: \mathbb{Z}^+ \to \mathbb{R}^+$, $f(n) \in \Theta(f(n))$ $\qquad$ $\forall n \geq 1,\ 1 \cdot f(n) \leq f(n) \leq 1 \cdot f(n)$

THM: For any functions $f, g: \mathbb{Z}^+ \to \mathbb{R}^+$, if $f(n) \in \Theta(g(n))$ then $g(n) \in \Theta(f(n))$

Proof: Let f, g be as given.

Then $\exists n_0 \in \mathbb{Z}^+, c_1, c_2 \in \mathbb{R}^+$ s.t. $\forall n \in \mathbb{Z}^+, n \geq n_0 \to c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ (def Θ)

Find such n_0, c_1, c_2; let $n \geq n_0$ and $n \in \mathbb{Z}^+$. Then $c_1 \cdot g(n) \leq f(n)$ and $f(n) \leq c_2 \cdot g(n)$ (universal modus ponens)

and $g(n) \leq \frac{1}{c_1} \cdot f(n)$ and $\frac{1}{c_2} f(n) \leq g(n)$ (dividing by positives)

so $\frac{1}{c_2} \cdot f(n) \leq g(n) \leq \frac{1}{c_1} \cdot f(n)$

$\therefore \forall n \in \mathbb{Z}^+, n \geq \underset{m_0}{n_0} \to \underbrace{\frac{1}{c_2}}_{d_1} \cdot f(n) \leq g(n) \leq \underbrace{\frac{1}{c_1}}_{d_2} \cdot f(n)$ (generic particular + conditional)

$\therefore \exists m_0 \in \mathbb{N}^+, d_1, d_2 \in \mathbb{R}^+$ s.t. $\forall n \in \mathbb{Z}^+, n \geq m_0 \to d_1 \cdot f(n) \leq g(n) \leq d_2 \cdot f(n)$ (example)

↳ namely $m_0 = n_0$, $d_1 = \frac{1}{c_2}$, $d_2 = \frac{1}{c_1}$

$\therefore g(n) \in \Theta(f(n))$ (def Θ)

THM: For any functions $f, g, h: \mathbb{Z}^+ \to \mathbb{R}^+$, if $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$ then $f(n) \in O(h(n))$

Proof: Let f, g, h be as given.

Then $\exists n_0, c$ s.t. $\forall n \geq n_0$, $f(n) \leq c \cdot g(n)$

and $\exists m_0, d$ s.t. $\forall n \geq m_0$, $g(n) \leq d \cdot h(n)$

[want $\exists k_0, e$ s.t. $\forall n \geq k_0$, $f(n) \leq e \cdot h(n)$]

Let $n \geq \max(n_0, m_0)$, $n \in \mathbb{Z}^+$.

Then $n \geq n_0$ and $n \geq m_0$ so $\frac{1}{c} \cdot f(n) \leq g(n) \leq d \cdot h(n)$

$\frac{1}{c} \cdot f(n) \leq d \cdot h(n)$

$f(n) \leq c \cdot d \cdot h(n)$

$\forall n \in \mathbb{Z}^+, n \geq \underset{k_0}{\max(n_0, m_0)} \to f(n) \leq \overset{e}{c \cdot d} \cdot h(n)$ $\qquad$ $e \in \mathbb{R}^+$ b/c $c, d \in \mathbb{R}^+$

$\exists k_0 \in \mathbb{Z}^+, e \in \mathbb{R}^+$ s.t. $\forall n \in \mathbb{Z}^+, n \geq \max(n_0, m_0) \to f(n) \leq c \cdot d \cdot h(n)$

$f(n) \in O(h(n))$

THM: For all $f, g, h: \mathbb{Z}^+ \to \mathbb{R}^+$, if $f(n) \in O(g(n))$ then $g(n) \in \Omega(f(n))$

if $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$ then $f(n) \in \Theta(g(n))$

if $f(n) \in \Theta(g(n))$ and $g(n) \in \Theta(h(n))$ then $f(n) \in \Theta(h(n))$

$f\,R\,g$ when $f(n) \in \Theta(g(n))$

equivalence relations

payoff for I

$$\begin{array}{ccccc} & & & \text{Player II} & \\ & & \text{Rock} & \text{Paper} & \text{Scissors} \\ & \text{Rock} & 0 & -1 & 1 \\ \text{Player} & \text{Paper} & 1 & 0 & -1 \\ \text{I} & \text{Scissors} & -1 & 1 & 0 \end{array}$$

P(I plays Rock) P(I plays paper)

$\frac{4}{10}$ $\frac{3}{10}$ $\frac{3}{10}$

P(I plays scissors)

dot product

Expected payoff for I when II plays Rock $= \frac{4}{10} \cdot 0 + \frac{3}{10} \cdot 1 + \frac{3}{10} \cdot -1 = 0$

Paper $= \frac{4}{10} \cdot -1 + \frac{3}{10} \cdot 0 + \frac{3}{10} \cdot 1 = -\frac{1}{10}$

Scissors $= \frac{4}{10} \cdot 1 + \frac{3}{10} \cdot -1 + \frac{3}{10} \cdot 0 = \frac{1}{10}$

$$\begin{pmatrix} \frac{4}{10} & \frac{3}{10} & \frac{3}{10} \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{10} & \frac{1}{10} \end{pmatrix}$$

vector (1-row matrix) of probabilities

payoff matrix

payoffs for each option for II

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2p} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & \cdots & c_p \end{pmatrix}$$

R P I

$1 \times n$ $\qquad$ $n \times p$ $\qquad$ $1 \times p$

1×3 $\qquad$ 3×3 $\qquad$ 1×3

where $c_j = \sum_{k=1}^{n} a_k \cdot b_{kj}$

m-row, n-col $\qquad$ n-row, p-col $\qquad$ m-row, p-col

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2p} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1p} \\ \vdots & & \vdots \\ c_{m1} & \cdots & c_{mp} \end{pmatrix}$$

where $\underline{c_{ij}} = \underbrace{\sum_{k=1}^{n} a_{ik} \cdot b_{kj}}_{\text{dot product of row } i \text{ of 1st matrix, col } j \text{ of 2nd matrix}}$

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 3 \\ 3 & 1 \end{pmatrix}$$

2×3 $\qquad$ 3×2

$2 \cdot 0 + -1 \cdot -1 + 0 \cdot 1$

$2 \cdot 2 + -1 \cdot 1 + 0 \cdot 2$

3×2 $\quad 2\cdot 2 + -1\cdot 1 + 0\cdot 2$

$$\begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

$3\times 2 \qquad 2\times 3 \qquad 3\times 3$

$$\begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 3 \\ 2 & -1 & 0 \\ 1 & -1 & 4 \end{pmatrix}$$

$3\times 2 \qquad 4\times 3$

$2\cdot 0 + 1\cdot 2$

$3\times 2 \cdot 2\times 4$

$2\times 4 \cdot 3\times 2$

$$\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & \\ & \end{pmatrix}$$

Matrix multiplication is not commutative not the case that $A\cdot B = B\cdot A$ always

I's prob. dist. over actions

II's prob dist

$$\begin{pmatrix} \frac{4}{10} & \frac{3}{10} & \frac{3}{10} \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{10} \\ \frac{4}{10} \\ \frac{3}{10} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{10} & \frac{1}{10} \end{pmatrix} \begin{pmatrix} \frac{3}{10} \\ \frac{4}{10} \\ \frac{3}{10} \end{pmatrix}$$

$$= -\frac{1}{100}$$

$$\begin{pmatrix} \frac{4}{10} & \frac{3}{10} & \frac{3}{10} \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{10} \\ 0 \\ \frac{1}{10} \end{pmatrix} = -\frac{1}{100}$$

Matrix multiplication is associative