

Crazy Function

Define $f: \mathbb{Z}^+ \rightarrow \mathbb{R}^+$ by $f(n) = \begin{cases} 2^n & \text{if } n \leq 100 \\ 2^{100} \cdot n^4 & \text{otherwise} \end{cases}$

- Is $f(n) \in O(2^n)$ Y $\forall n \geq 101, 2^{100} \cdot n^4 \leq 2^{100} \cdot 2^n$
 $f(n) \in \Theta(2^n)$ N $\exists n_0, c, \forall n \geq n_0, f(n) \leq c \cdot 2^n$ b/c for $n \geq 101$
 $f(n) \in O(n^4)$ Y $f(n) \in O(2^n)$ $n^4 \leq 2^n$
 $f(n) \in \Theta(n^4)$ Y $\forall n \geq 101, \frac{f(n)}{n^4} = 2^{100} \leq 2^{100} \cdot n^4$

$$2^{100} n^4 \in \Theta(n^4) \quad n^4 \in \Theta(2^{100} \cdot n^4)$$

THM: For any function $f: \mathbb{Z}^+ \rightarrow \mathbb{R}^+$, $f(n) \in \Theta(f(n))$ $\forall n \geq 1, 1 \cdot f(n) \leq f(n) \leq 1 \cdot f(n)$

THM: For any functions $f, g: \mathbb{Z}^+ \rightarrow \mathbb{R}^+$, if $f(n) \in \Theta(g(n))$ then $g(n) \in \Theta(f(n))$

Proof: Let f, g be as given.

Then $\exists n_0 \in \mathbb{Z}^+, c_1, c_2 \in \mathbb{R}^+$ s.t. $\forall n \in \mathbb{Z}^+, n \geq n_0 \rightarrow c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ (def Θ)

Find such n_0, c_1, c_2 ; let $n \geq n_0$. Then $c_1 \cdot g(n) \leq f(n)$ and $f(n) \leq c_2 \cdot g(n)$ (universal modus ponens)

and $g(n) \leq \frac{1}{c_1} \cdot f(n)$ and $\frac{1}{c_2} \cdot f(n) \leq g(n)$ (dividing by positives)

so $\frac{1}{c_2} \cdot f(n) \leq g(n) \leq \frac{1}{c_1} \cdot f(n)$

$\therefore \forall n \in \mathbb{Z}^+, n \geq n_0 \rightarrow \frac{1}{c_2} \cdot f(n) \leq g(n) \leq \frac{1}{c_1} \cdot f(n)$ (general particular conditional)

$\therefore \exists m_0, d_1, d_2$ s.t. $\forall n \in \mathbb{Z}^+, n \geq m_0 \rightarrow d_1 \cdot f(n) \leq g(n) \leq d_2 \cdot f(n)$ (example)

\hookrightarrow namely $m_0 = n_0, d_1 = \frac{1}{c_2}, d_2 = \frac{1}{c_1}$

$\therefore g(n) \in \Theta(f(n))$ (def Θ)

THM: For any functions $f, g, h: \mathbb{Z}^+ \rightarrow \mathbb{R}^+$, if $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$ then $f(n) \in O(h(n))$

Proof: Let f, g, h be as given.

Then $\exists n_0, c$ s.t. $\forall n \geq n_0, f(n) \leq c \cdot g(n)$
and $\exists m_0, d$ s.t. $\forall n \geq m_0, g(n) \leq d \cdot h(n)$

Let $n \in \mathbb{Z}^+$
Let $n \geq \max(n_0, m_0)$.

[want $\exists k_0, e$ s.t. $\forall n \geq k_0, f(n) \leq e \cdot h(n)$]

Then $n \geq n_0$ and $n \geq m_0$ so $\frac{1}{c} \cdot f(n) \leq g(n) \leq d \cdot h(n)$
 $\frac{1}{c} \cdot f(n) \leq d \cdot h(n)$

$\forall n \in \mathbb{Z}^+, n \geq \max(n_0, m_0) \rightarrow f(n) \leq c \cdot d \cdot h(n)$ $e \in \mathbb{R}^+ \text{ b/c } c, d \in \mathbb{R}^+$

$\exists k_0 \in \mathbb{Z}^+, e \in \mathbb{R}^+$ s.t. $\forall n \in \mathbb{Z}^+, n \geq \max(n_0, m_0) \rightarrow f(n) \leq e \cdot h(n)$
 $f(n) \in O(h(n))$

THM: For all $f, g, h: \mathbb{Z}^+ \rightarrow \mathbb{R}^+$, if $f(n) \in O(g(n))$ then $g(n) \in \Omega(f(n))$

if $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$
then $f(n) \in \Theta(g(n))$

if $f(n) \in \Theta(g(n))$ and $g(n) \in \Theta(h(n))$
then $f(n) \in \Theta(h(n))$

$f \sim g$ when $f(n) \in \Theta(g(n))$
equivalence relations

Simultaneous Games

		Player II		
		Rock	Paper	Scissors
Player I	Rock	0	-1	1
	Paper	1	0	-1
	Scissors	-1	1	0

$P(I \text{ plays Rock}) = \frac{4}{10}$
 $P(I \text{ plays paper}) = \frac{3}{10}$
 $P(I \text{ plays scissors}) = \frac{3}{10}$

payoff for I
 dot product

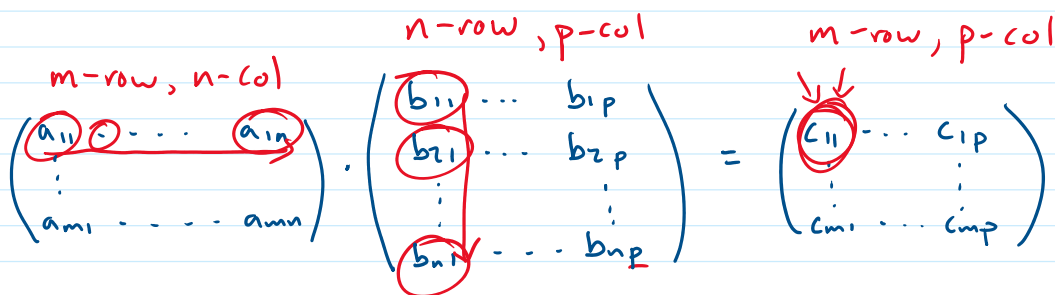
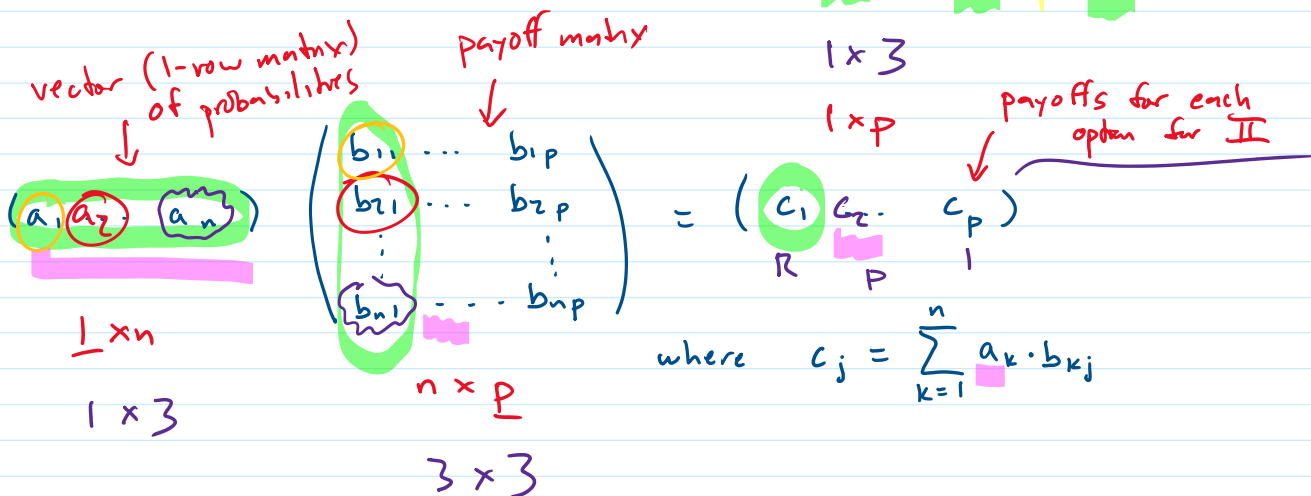
Expected payoff for I when II plays

Rock = $\frac{4}{10} \cdot 0 + \frac{3}{10} \cdot 1 + \frac{3}{10} \cdot (-1) = 0$

Paper = $\frac{4}{10} \cdot 1 + \frac{3}{10} \cdot 0 + \frac{3}{10} \cdot (-1) = -\frac{1}{10}$

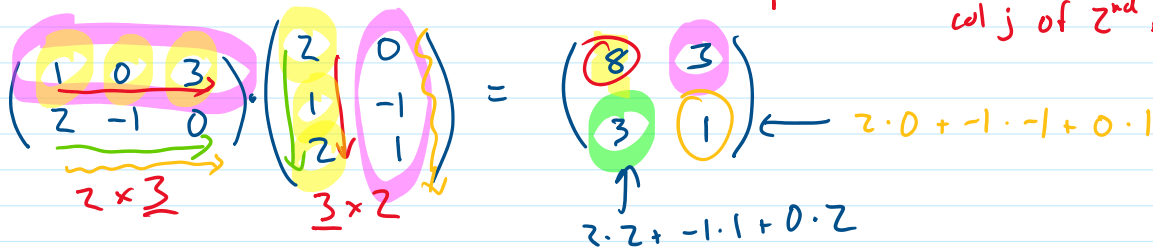
Scissors = $\frac{4}{10} \cdot (-1) + \frac{3}{10} \cdot 1 + \frac{3}{10} \cdot 0 = \frac{1}{10}$

$$\left(\frac{4}{10} \quad \frac{3}{10} \quad \frac{3}{10} \right) \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} = \left(0 \quad -\frac{1}{10} \quad \frac{1}{10} \right)$$



where $c_{ij} = \sum_{k=1}^n a_{ik} \cdot b_{kj}$

dot product of row i of 1st matrix and col j of 2nd matrix



2×2

$2 \cdot 2 + -1 \cdot 1 + 0 \cdot 2$

$$\begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

3×2 2×3 3×3

~~$$\begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 3 \\ 2 & -1 & 0 \\ 1 & -1 & 4 \end{pmatrix}$$

3×2 4×3~~

$2 \cdot 0 + 1 \cdot 2$

$3 \times 2 \cdot 2 \times 4$

$2 \times 4 \cdot 3 \times 2$

$$\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & \\ & \end{pmatrix}$$

Matrix multiplication is not commutative not the case that $A \cdot B = B \cdot A$ always

I's prob. dist. over actions

II's prob. dist

$$\begin{pmatrix} \frac{4}{10} & \frac{3}{10} & \frac{3}{10} \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{10} \\ \frac{4}{10} \\ \frac{1}{10} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{10} & \frac{1}{10} \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{10} \\ \frac{4}{10} \\ \frac{1}{10} \end{pmatrix}$$

$$= -\frac{1}{100}$$

$$\begin{pmatrix} \frac{4}{10} & \frac{3}{10} & \frac{3}{10} \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{10} \\ 0 \\ \frac{1}{10} \end{pmatrix} = -\frac{1}{100}$$

Matrix multiplication is associative