Taste the Rainbow

$$\underset{x_1}{\_\_} + \underset{x_2}{\_\_} + \underset{x_3}{\_\_} + \underset{x_4}{\_\_} + \underset{x_5}{\_\_} = 3$$

cherry   # grape   # lime   # orange   # lemon

$$0 \le x_i \le 3$$

$$0 + 0 + 0 + 1 + 2 = 3$$

$$0 + 0 + 0 + 0 + 3 = 3$$

$$0 + 0 + 1 + 1 + 1 = 3$$

strings of 7 1's and +'s w/ 4 +'s and 3 1's

+ + + | + || = 3

+ + + + |||

+ + | + | + |

total # of such strings = total ways of choosing $x_1, \ldots, x_5$ = $\binom{7}{3} = \binom{7}{4}$

$$\binom{n}{k} = \binom{n}{n-k}$$

Define R on $\mathcal{P}(\mathbb{N})$ by ARB iff $\underline{A \subseteq B}$

reflexive YES $\forall A \subseteq \mathbb{N}$, $A \subseteq A$

$A \subseteq B$ and $B \subseteq A$ only when $A = B$

symmetric NO <u>antisymmetric</u> $\forall a, b \in A \to aRb \wedge a \neq b \to b \not R a$

transitive YES $\forall A, B, C \subseteq \mathbb{N}$, $A \subseteq B \wedge B \subseteq C \to A \subseteq C$

$a \preceq b \wedge a \neq b$

$\sim(b \preceq a)$

DEF: A partial order $\preceq$ on a set S is a relation on S that is reflexive, antisymmetric, transitive

$\subseteq$ is a <u>partial</u> order

$\{0, 1, 2\} \subseteq \{2, 3, 4\}$? NO

$\{2, 3, 4\} \subseteq \{0, 1, 2\}$? NO

incomparable

Define $\leq$ on $\mathbb{R}$ by $a \leq b$ iff $a \leq b$

reflexive $\forall a \in \mathbb{R}$ $a \leq a$

transitive $\forall a, b, c \in \mathbb{R}$ $a \leq b \wedge b \leq c \to a \leq c$

antisymmetric $\forall a, b \in \mathbb{R}$ $a \leq b \wedge a \neq b \to \sim(b \leq a)$

DEF: A <u>total order</u> $\leq$ on set A is a partial order on A such that for all $\underline{a, b \in A}$, aRb or bRa

$\leq$ on $\mathbb{R}$ is total order b/c $\forall a, b \in \mathbb{R}$ $a \leq b$ or $b \leq a$

DEF: For a set A with partial order $\preceq$,

a is a maximal element means there is no $b \in A$ s.t. $a \preceq b$ and $b \neq a$

a is the greatest element means for all $b \in A$ $b \preceq a$ or $b = a$

a is a minimal element means there is no $b \in A$ s.t. $b \preceq a$ and $b \neq a$

a is the least element means for all $b \in A$ $a \preceq b$ or $b = a$

$\subseteq$ on $\mathcal{P}(\mathbb{N})$ maximal $\mathbb{N}$ minimal $\emptyset$

$\underline{\mathbb{N}} \subseteq \mathbb{N}$

$\underline{\emptyset} \subseteq \emptyset$

$\leq$ on $\mathbb{N}$ maximal none!

$10^{100} \leq \underline{10^{100} + 1}$

$\leq$ on $\mathbb{N}$

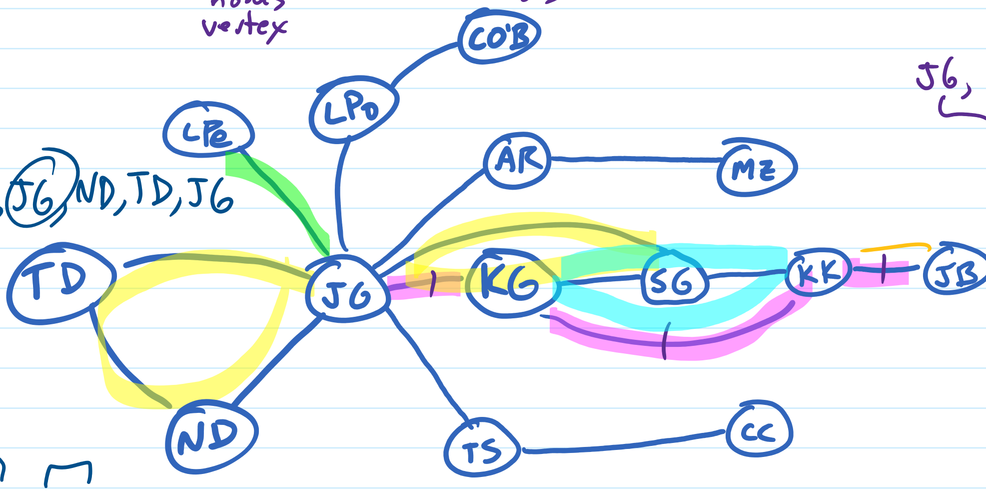
maximal none!

minimal 0 $0 \leq 0$ $10^{100} \leq 10^{100}+1$

Graphs

↪ representation of things (people; nodes, vertex) and relationships (relationships; edges) between them

JG, KG, SG, JG, ND, TD, JG



JG, KG, KK, JB

length = 3

MZ

length 0

path

KG, SG, KK, KG

SG, KK, KG, SG

JG, KG, JG

simple graph: no self-loops, no parallel edges

path: non-empty seq. of verts s.t. edge between adj verts in seq

simple path: path w/ no repeated verts

cycle: path of length ≥ 3 beginning/ending at same place
- no repeated edges
- only repeated vertex is 1st/last

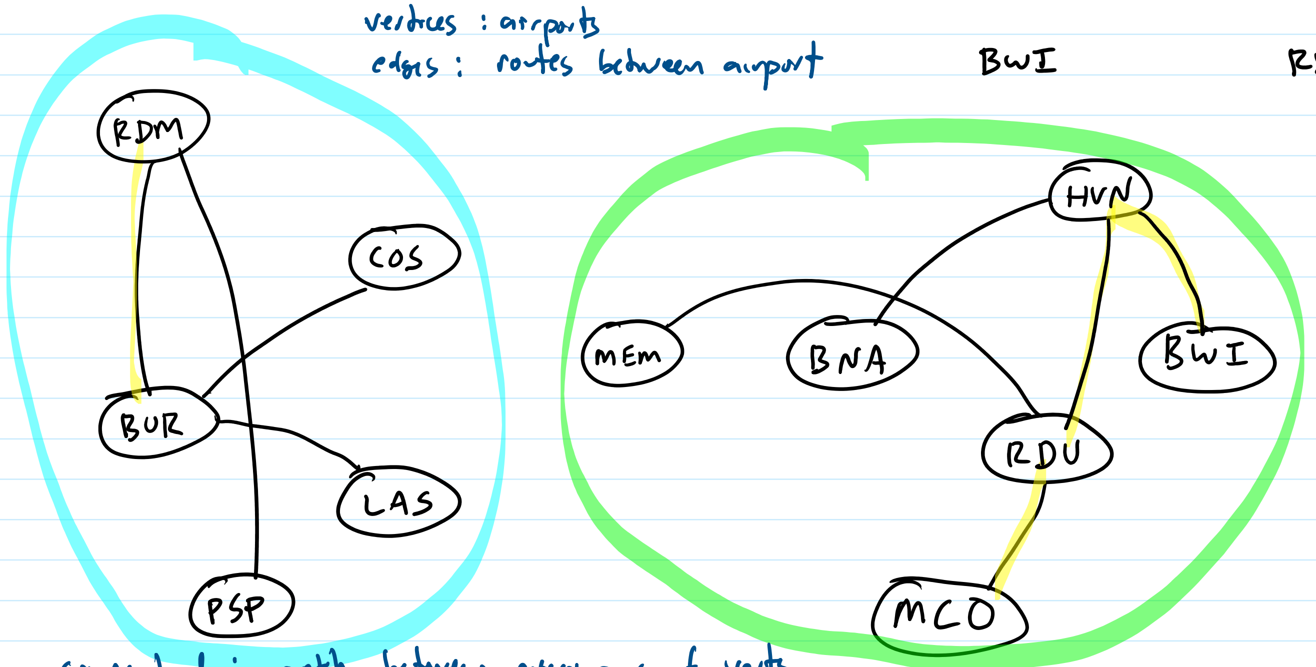
circuit: cycle w/ only 1st/last repeat relaxed

closed walk: path w/ same start/end

Airline Route Map

vertices : airports
edges : routes between airport

BWI RDM

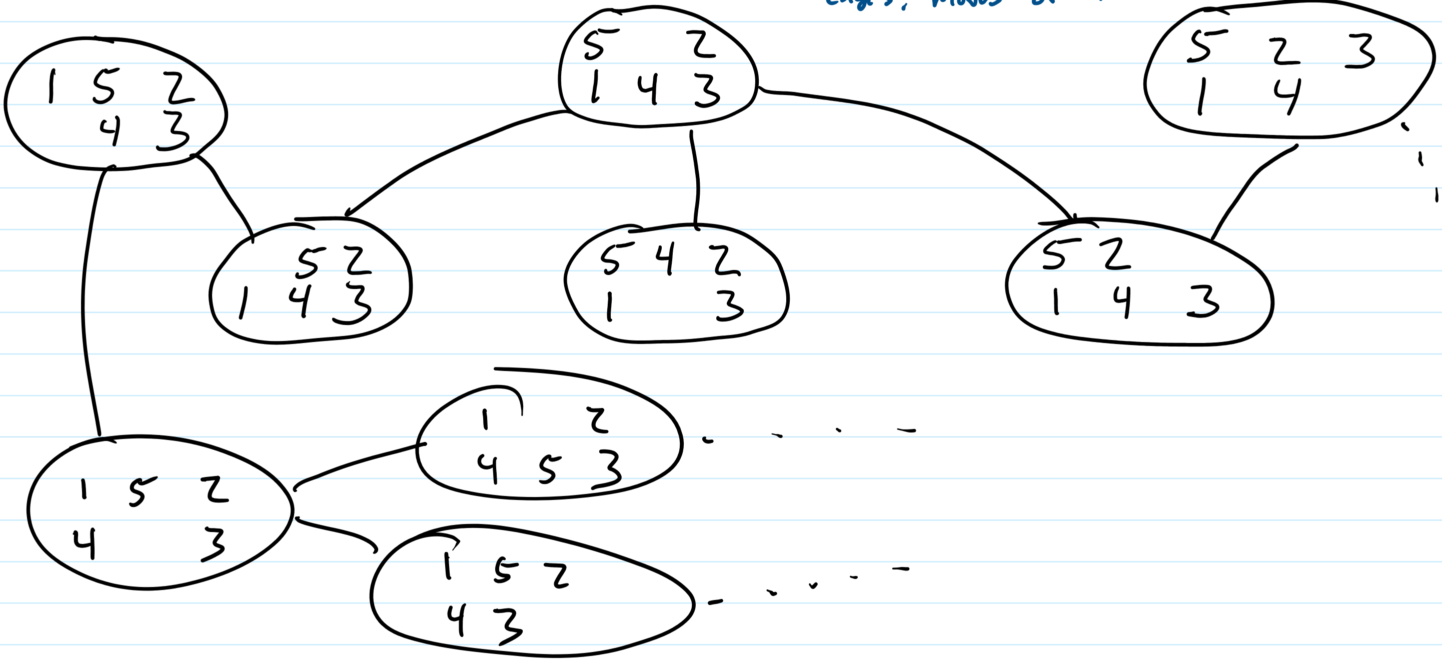


connected : path between every pair of verts

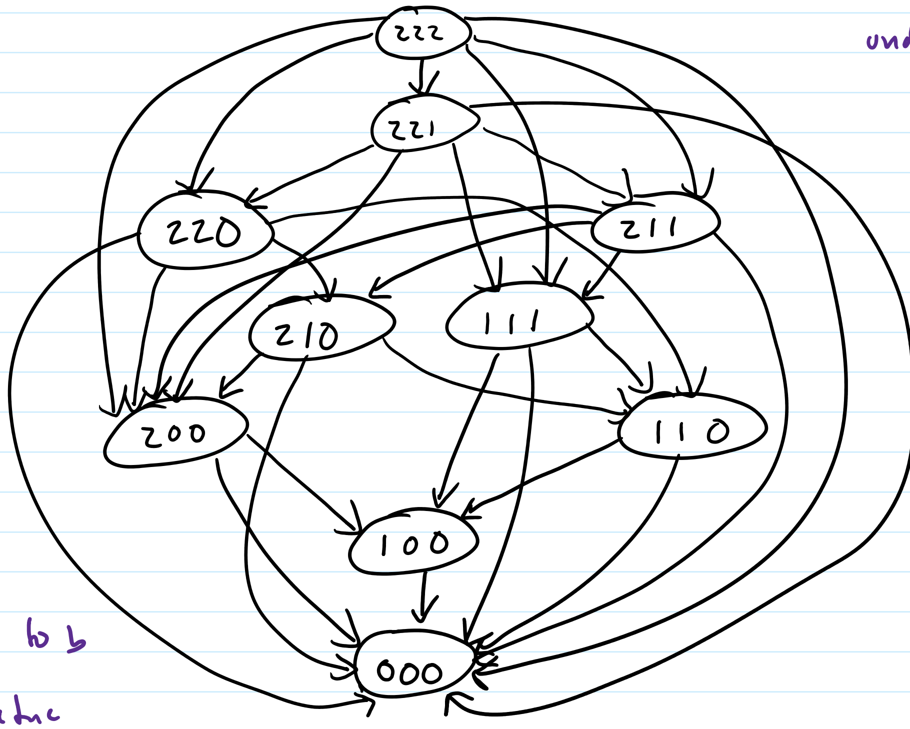
connected component : maximal subset of verts that is connected

5 puzzle

vertices: arrangements of tiles
edges: moves of 1 tile



Chomp



undirected : irreflexive
symmetric

a

Edge a → b

pick square in a
to leave state b

aRb iff
can move from a to b

chomp antisymmetric

Paths

$n = n(V)$ $m = n(E)$

THM: If u, v connected in a graph $G = (V, E)$, then there is a path $u \rightsquigarrow v$ of length at most $n-1$.

Proof: Let $G = (V, E)$ be a graph and suppose $u, v \in V$ are connected.

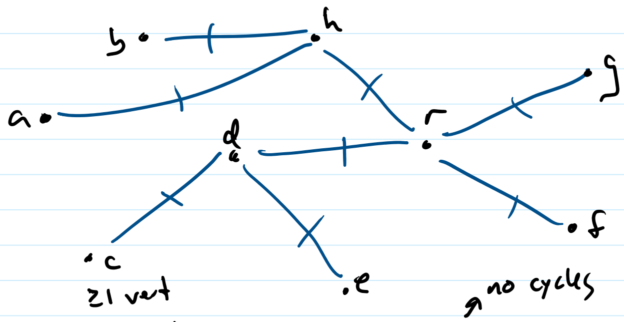
Then there is a shortest path $v_0, \dots, v_k$ in G with $v_0 = u$ and $v_k = v$.

Two cases: 1) $k \leq n-1$

2) $k > n-1$

Tree

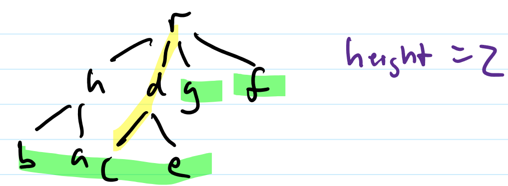
↳ connected, acyclic, undirected graph



THM: Any non-empty, connected, acyclic, undirected graph has n-1 edges

THM: In a tree, there is a unique simple path between any pair of verts

Rooted tree: tree w/ distinguished node (root)

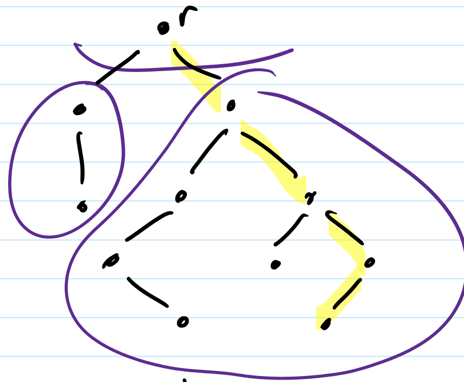


parent of c = d

leaf: no children

Binary Tree

Binary tree: a rooted tree in which every node has ≤ 2 children



height

THM: All binary trees have $\le 2^{h+1}-1$ nodes

Proof: We prove for all possible heights h, all binary trees of height h have $\le 2^{h+1}-1$ nodes

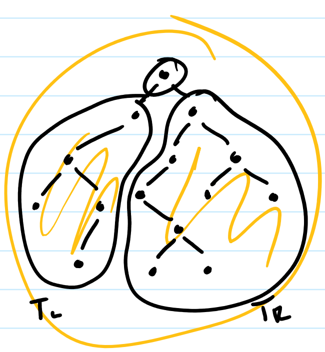
Base cases:

$(h=0)$ one node $2^{h+1}-1 = 1$; $1 \le 1$

Ind step: suppose $h > 0$ and for all i, $0 \le i < h$, all trees of height i have $\le 2^{i+1}-1$ nodes

Let T be a tree of height h

$h_L, h_r \le h-1$



$$n_L \le 2^{h_L+1}-1 \le 2^h-1$$

$$n_R \le 2^{h_R+1}-1 \le 2^h-1$$

$$n = n_L + n_r + 1 \le 2^h-1 + 2^h-1 + 1$$

$$= 2^{h+1}-1$$