Taste the Rainbow

$n=5$
$r=3$

strings of 0s 1s of length 7
3 0's 4 1's

| |000| |

|0|0| |0

|0|0|0|

5 bins →
4 dividers between

choosing r things
n choices

$$\binom{n-1+r}{r}$$

1) choose where 0's go $\binom{7}{3} = \frac{7!}{3!\cdot 4!} = \frac{7\cdot 6\cdot 5\cdot \cancel{4}}{\cancel{4!}} = 35$
2) put 1's everywhere else 1

1) strings of r 0's $\binom{n-1+r}{r}$
2) and $n-1$ 1's 1

order

repetition		matters (permutations)	doesn't (combinations)
	no	$\frac{n!}{(n-r)!}$	$\binom{n}{r} = \frac{n!}{(n-r)!\cdot r!}$
	yes	strings n^r	multisets $\binom{n+r-1}{r}$

n = # things to choose from
r = # things to choose

$\{1,1,3\} = \{3,1,1\} \neq \{1,3\}$

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7 flavors
choose 4

$n=7$
$r=4$

$$\binom{7+4-1}{3} = \binom{10}{4} = \frac{10!}{6!\cdot 4!} = \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2}$$

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$$\frac{n(E)}{n(S)}$$

$$P(\text{roll a 5 of kind}) \neq \frac{\#\text{ multisets that are 5K}}{\#\text{ multisets total}} = \frac{6}{252}$$

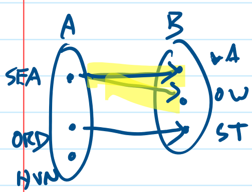
~~11112~~ more likely than 11111

Relations

function: set $\{(x, f(x)) \mid x \in \text{Domain}\}$

DEF: A relation R from A to B is a subset of $A \times B$

Example: $A = \{\text{SEA}, \text{HVN}, \text{ORD}\}$ $B = \{*A, \text{OW}, \text{ST}\}$



$R = \{(\text{SEA}, \text{OW}), (\text{SEA}, *A), (\text{ORD}, \text{ST})\}$

write aRb for $(a,b) \in R$

$A = \{a, b, c\}^*$, $B = \mathbb{N}$ $R = \{(a,b) \in A \times B \mid \exists x \in \{a,b,c\}$ s.t. a contains exactly b occurrences of $x\}$

Let aRb mean there is a letter in a that appears exactly b times

$\varepsilon R\, 0$ $\varepsilon \not R\, 1$ $\sim(\varepsilon R 1)$

$cab\ R\ 1$ $cab \not R\ 2$ $cab \not R\ 0$

relation from A to A

DEF: A relation R on A is a subset of $A \times A$

Ex: Define relation R on $\mathbb{R}$ by aRb if and only if $a = b^2$

$0R0$, $0 = 0^2$ $9R3$, $9 = 3^2$ $9R{-3}$, $9 = (-3)^2$ $\sqrt{2}\ R\ {+\sqrt{\sqrt{2}}}$ $+\sqrt{2}\ R\ {-\sqrt{\sqrt{2}}}$

DEF: A relation R on set A is

reflexive if and only if $\forall a \in A,\ aRa$

symmetric if and only if $\forall a, b \in A,\ aRb \to bRa$

transitive if and only if $\forall a, b, c \in A,\ aRb \wedge bRc \to aRc$

Ex: the square root relation is none of these
- not reflexive $\sim(2R2)$ $2 \neq 2^2$
- not symmetric $9R3$ but $\sim(3R9)$
- not transitive $16R4 \wedge 4R2$ but $\sim(16R2)$

Define R on $\mathbb{Z}$ by aRb iff $a \cdot b > 0$
- reflexive NO $\sim(0R0)$
- symmetric YES
- transitive YES

aRb iff $a \cdot b \geq 0$

for some $n \geq 2$

aRb iff $a \equiv b \pmod{n}$

reflexive? YES
symmetric? YES
transitive? YES

$\exists k$ s.t. $b-a=nk$
$n \mid b-a$
then $n \mid a-b$

DEF: A relation R on A is an equivalence relation means R is reflexive, symmetric, transitive

DEF: If R is an equivalence relation on set A, the equivalence class of $x \in A$ is $[x] = \{ y \in A \mid yRx \}$

the set of things related to x

$y \equiv 0 \pmod 3$

$3 \mid y - 0$

for $\equiv_3$ the equivalence classes are

$\langle 3 \rangle =$

$[0] = \{ \ldots, -3, 0, 3, 6, 9, \ldots \}$

$a R b$ iff $a \equiv b \pmod 3$

$[1] = \{ \ldots, -2, 1, 4, 7, 10, \ldots \}$

$3 \mid y - 1$

$y \equiv 1 \pmod 3$

$[2] = \{ \ldots, -1, 2, 5, 8, 11, \ldots \}$

"has the same cardinality as" equiv classes

equivalence relation

$[\emptyset] = \{ \emptyset \}$

$[\{1\}] = \{ \{1\}, \{2\}, \{a\}, \{\infty\}, \{8\}, \ldots \}$

continuum hypothesis

are there sets w/ cardinality between?

$[\mathbb{N}] = \{ \mathbb{N}, \mathbb{Z}, \mathbb{N} \times \mathbb{N}, 2\mathbb{Z}, 10\mathbb{Z}, \ldots \}$

$[\mathbb{R}] = \{ \mathbb{R}, (0,1), [0,1], \ldots \}$

THM: For any equivalence relation R on any set A,

1) For every $a \in A$, there is a $b \in A$ s.t. $a \in [b]$

aRa $\quad a \in [a]$

2) For every $a, b \in A$, aRb if and only if $[a] = [b]$

Proof:

$\Leftarrow$

3) For every $a, b \in A$, either $[a] \cap [b] = \emptyset$ or $[a] = [b]$

Proof:

The equivalence classes induced by equivalence relation R on A partition A

Define R on $\mathcal{P}(\mathbb{N})$ by ARB iff $\underline{A \subseteq B}$

reflexive YES $\forall A \subseteq \mathbb{N}$, $A \subseteq A$

$A \subseteq B$ and $B \subseteq A$ only when $A = B$

symmetric NO $\underline{\text{antisymmetric}}$ $\forall a, b \in A \rightarrow aRb \wedge a \neq b \rightarrow b \not R a$

transitive YES $\forall A, B, C \subseteq \mathbb{N}$, $A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C$

DEF: A partial order $\preceq$ on a set S is a relation on S that is reflexive, antisymmetric, transitive

$\subseteq$ is a $\underline{\text{partial}}$ order

$\{\underline{0}, 1, 2\} \subseteq \{2, 3, 4\}$? NO

$\{2, 3, 4\} \subseteq \{0, 1, 2\}$? NO

incomparable

Define $\leq$ on $\mathbb{R}$ by $a \leq b$ iff $a \leq b$

reflexive $\forall a \in \mathbb{R}$ $a \leq a$

transitive $\forall a, b, c \in \mathbb{R}$ $a \leq b \wedge b \leq c \rightarrow a \leq c$

antisymmetric $\forall a, b \in \mathbb{R}$ $a \leq b \wedge a \neq b \rightarrow \sim(b \leq a)$

DEF: A $\underline{\text{total order}}$ $\leq$ on set A is a partial order on A such that for all $\underline{a, b \in A}$, aRb or bRa

$\leq$ on $\mathbb{R}$ is total order b/c $\forall a, b \in \mathbb{R}$ $a \leq b$ or $b \leq a$

DEF: For a set A with partial order $\preceq$,

a is a maximal element means there is no $b \in A$ s.t. $a \preceq b$ and $b \neq a$

a is the greatest element means for all $b \in A$ $b \preceq a$ or $b = a$

a is a minimal element means there is no $b \in A$ s.t. $b \preceq a$ and $b \neq a$

a is the least element means for all $b \in A$ $a \preceq b$ or $b = a$

$\subseteq$ on $\mathcal{P}(\mathbb{N})$ maximal

minimal

$\leq$ on $\mathbb{N}$ maximal

minimal