Enumerative Combinatorics — the study of how to count things

Clique: Given n people and k, determine if there is a size-k subset who all know each other

$n = 24$ 100
$k = 4$ 10

→ for each possible size-k subset
    for all possible pairs in the subset
        ask if the pair knows each other
    if all answers were "YES", output the subset
output "NONE"

Chopsticks

P1 left	P1 right	P2 left	P2 right
1	1	1	1
1	1	1	2
2	1	1	2

⋮

how many different game states?

Probability

sample space S: set of all possible outcome

event E: subset of sample space

S = result of drawing 1 card
= {2C, 3C, ..., AC, 2D, ..., AD, 2S, ..., AS, 2H, ..., AH}

E: draw black face card
= {JC, QC, KC, JS, QS, KS}

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If all events in finite sample space S are equally likely, then

$$P(E) = \frac{n(E)}{n(S)}$$

$P(\text{draw black face card}) = \frac{6}{52} = \frac{3}{26}$

S = outcome of flipping 2 fair coins
E = exactly one head

S = {0 heads, 1 head, 2 heads}

E = {1 head}

~~P(flipping 1 head) = $\frac{1}{3}$~~

1980 penny

1997 penny

S = {HH, HT, TH, TT}

E = {HT, TH}

$P(\text{flipping 1 head}) = \frac{2}{4} = \frac{1}{2}$

Multiplication Rule

If a k-step process has $n_1, \dots, n_k$ possible choices at each step, and all the possible outcomes are distinct, then the number of possible outcomes is

Ex: $A = \{1, 2, 3\}$ $B = \{a, b, c, d\}$

$n(A \times B) = 3 \cdot 4 = 12$ — 1) choose 1st component 3 — 2) choose 2nd component 4 — $3 \cdot 4$

$(\_\_ , \_\_)$ — 1, 2, or 3: 3 choices — a, b, c, or d: 4 choices

$\{(1,a), (1,b), (1,c), (1,d), (2,a), \dots \dots (3,d)\}$

$f(1) = c$, $f(2) = a$, $f(3) = d$

$f(1) = d$, $f(2) = d$, $f(3) = d$

Ex: $F = \{ f \mid f: A \to B \}$

$n(F) = 4 \cdot 4 \cdot 4 = 4^3 = 64$

1) choose $f(1)$ 4
2) choose $f(2)$ 4
3) choose $f(3)$ 4

$n(A) = c$
$n(B) = d$

functions $f: A \to B = d^c$

Ex: $X = \{x_1, x_2, \dots, x_n\}$

$n(\mathcal{P}(X)) = 2^n$

$\{x_1, x_4, \dots\}$

to choose an elt of $\mathcal{P}(X)$ (a subset of X)

1) determine whether x_1 is in or not 2
2) " " x_2 " 2
⋮ ⋮
n) " " x_n " 2

Rewards code: 14 characters, 32 possible (A-Z, 2-9) (I and O)

$$P(\text{randomly chosen code is valid}) = \frac{n(\text{valid codes})}{n(\text{sample space})} \approx \frac{10^6}{10^{21}} = \frac{1}{10^{15}}$$

31M years at 1/sec

1) choose 1st char 32
2) choose 2nd char 32
⋮ ⋮
14) choose 14th char 32

total # sequences of choices $= 32^{14} = 2^{70} \approx 10^{21}$

$P(\text{both fair 6-sided dice are} \le 3)$

$S = \{11, \dots, 66\}$

$E = \{11, 12, 13, 21, \dots, 33\}$

Mechanical Lock: n buttons, must push all in correct order to unlock

how many possible 4 digits combinations w/ 6 buttons?

w (no repetition)

1) pick 1st digit — 6
2) pick 2nd digit — 5 (every digit except one chosen in step 1)
3) pi 3rd — 4
4) pick 4th — 3

$6 \cdot 5 \cdot 4 \cdot 3 = 360$

P(roll a straight with 6 6-sided dice)

$\underbrace{\text{roll a straight with 6 6-sided dice}}_{E}$

$S = \{111111, \dots, 666666\}$ 211111   121111

$n(S) = 6^6$

1) pick 1st thing — 6
2) pick 2nd thing — 5
⋮ — 4
6) picking last — 1

$\to E = \{123456, \dots, 654321\}$

$n(E) = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6!$ 6 factorial

$P(\text{rolling straight}) = \dfrac{6!}{6^6}$

permutation: ordering of things

Number of permutations of n things: $n!$

1) pick 1st — n
⋮
n) pick last — 1

product $= n(n-1)(n-2) \cdot \dots \cdot 1 = n!$

r-permutation: an ordering of a size-r subset

1) pick 1st — n
— $n-1$
⋮
r) pick rth — $n-r+1$

product $\dfrac{n!}{(n-r)!}$

10-button mechanical lock, 4-digit code

$\#$ possible codes $= \#$ 4-permutations of 10 digits $= \dfrac{10!}{(10-4)!} = 10 \cdot 9 \cdot 8 \cdot 7$

P(roll a total of ≤ 4 with 2 fair 6-sided dice)