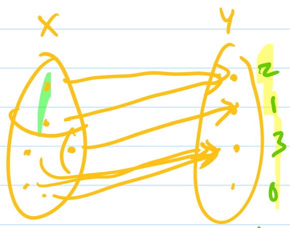
Addition Rule: for finite disjoint sets $X_1, \dots, X_m$

$$n\left(\bigcup_{i=1}^{m} X_i\right) = \sum_{i=1}^{m} n(X_i)$$



$n(X) = 2+1+3+0 = 6$

Ex: $A = \{1, 2, \dots, k\}$ $\quad B = \{(a,b) \in A \times A \mid a < b\}$

$= \{(1,2), (1,3), \dots, (1,k)$
$(2,3), (2,4), \dots, (2,k)$
$\vdots$
$(k-1,k)\}$

$n(B) = ??$

$B_1 = \{(a,b) \in B \mid a = 1\}$

$B_2 = \{(a,b) \in B \mid a = 2\}$

$\vdots$

$B_k = \{(a,b) \in B \mid a = k\}$

$B = \bigcup_{i=1}^{k} B_i$

if $k = 10$

$B_1 = \{(1,2), (1,3), (1,4), \dots, (1,10)\}$

$n(B_1) = 9$

for general k, $n(B_1) = k-1$

$n(B_2) = k-2$

$\vdots$

$n(B_k) = 0$

$$n(B) = \sum_{i=1}^{k} n(B_i) = \sum_{i=1}^{k} k - i = \sum_{i=1}^{k} k - \sum_{i=1}^{k} i$$

$$k^2 - \frac{k(k+1)}{2}$$

$$= \frac{k(k-1)}{2}$$

If a k-step process has $n_1, \dots, n_k$ possible choices at each step, and all the possible outcomes are distinct, then the number of possible outcomes is

$(\_\_ , \_\_)$

1, 2, or 3
3 choices

a, b, c, or d
4 choices

Ex: $A = \{1, 2, 3\}$ $B = \{a, b, c, d\}$

$\{(1,a), (1,b), (1,c), (1,d),$
$(2,a), \dots$
$\dots (3,d)\}$

$n(A \times B) = 3 \cdot 4 = 12$ – 1) choose 1st component 3
2) choose 2nd component 4

$3 \cdot 4$

$f(1) = c$ $f(1) = d$
$f(2) = a$ $f(2) = d$
$f(3) = d$ $f(3) = d$

Ex: $F = \{ f \mid f: A \to B \}$

$n(F) = 4 \cdot 4 \cdot 4$

1) choose $f(1)$ 4
2) choose $f(2)$ 4
3) choose $f(3)$ 4

$n(A) = c$
$n(B) = d$

$= 4^3$

functions $f: A \to B = d^c$

$= 64$

Ex: $X = \{x_1, x_2, \dots, x_n\}$

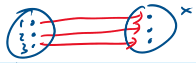
$n(\mathcal{P}(X)) = 2^n$

$\{x_1, x_4, \dots\}$

to choose an elt of $\mathcal{P}(X)$ (a subset of X)

1) determine whether x_1 is in or not 2
2) " " x_2 " 2
⋮ 2
n) " " x_n " 2

Cardinality



$n(X)=3$

DEF: A set X is finite means $\exists m \in \mathbb{N}$ s.t. there is a bijection $f:\{1,\dots,m\}\to X$

THM: For finite sets X, Y, $n(X)=n(Y)$ if and only if there is a bijection $f: X\to Y$

$\Rightarrow$ $\quad \{1,\dots,m\} \xrightarrow{f} X$, $\quad \{1,\dots,m\} \xrightarrow{g} Y$, $\quad X \xrightarrow{f^{-1}} \{1,\dots,m\}$

$g\circ f^{-1}$ — g: bijection, f^{-1}: bijection, so comp is bijection

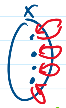
$\Leftarrow$ Pigeonhole

$|X|=|Y|$

DEF: For any sets X, Y, X has the same cardinality as Y means there is a bijection $f: X\to Y$

equivalence relations

THM: For any set X, $|X|=|X|$ (reflexive) — bijection i_X



For any sets X,Y, $|X|=|Y| \to |Y|=|X|$ (symmetric)

suppose $f: X\to Y$ is a bijection, then f^{-1} is a bijection $Y\to X$

For any sets X,Y,Z, $|X|=|Y|$ and $|Y|=|Z| \to |X|=|Z|$ (transitive)

bijection $f: X\to Y$, bijection $g: Y\to Z$; $g\circ f$ is a bijection $X\to Z$

Let $E=\{x\in\mathbb{Z} \mid x \text{ is even}\}$ ($2\mathbb{Z}$) $\quad O=\{x\in\mathbb{Z}\mid x\text{ is odd}\}$ $\quad T=\{x\in\mathbb{Z}\mid \exists y\in\mathbb{Z} \text{ s.t. } x=3y\}$ ($3\mathbb{Z}$)

THM: $|E|=|O|$

E	...	-4	-2	0	2	4	6	...	let $f(n)=n+1$
O	...	-3	-1	1	3	5	7	...	

Proof: Define $f: E\to O$ by $f(n)=n+1$. Then f is 1-1 and onto

THM: $|2\mathbb{Z}|=|3\mathbb{Z}|$

$2\mathbb{Z}$	...	-4	-2	0	2	4	6	...
$3\mathbb{Z}$	...	-6	-3	0	3	6	9	...

Proof: Define $f: 2\mathbb{Z}\to 3\mathbb{Z}$ by $f(n)=\frac{3}{2}n$ $\quad f$ 1-1 & onto

1	-1
3	-2
5	-3

THM: $|\mathbb{N}|=|\mathbb{Z}|$
and $|\mathbb{Z}|=|\mathbb{N}|$
$\mathbb{Z}$ is countably infinite

$\mathbb{N}$	0	1	2	3	4	...
$\mathbb{Z}$	... -1	0	1	2	3	4 ...

Proof: define $f:\mathbb{N}\to\mathbb{Z}$ by $f(n)=\begin{cases}\frac{n}{2} & \text{if } n \text{ is even}\\ -\frac{n+1}{2} & \text{if } n \text{ is odd}\end{cases}$ $\quad f$ is 1-1 and onto

DEF: A set X is countably infinite means $|\mathbb{N}|=|X|$

DEF: A set X is countable means X is finite or countably infinite

DEF: A set X is uncountable means X is not countable

countably

$\mathbb{N}$	0	1	2	3	4	...
$-$non	0	2	4	6	8	...

THM: $\mathbb{N}\times\mathbb{N}$ is countably infinite

($\exists$ 1-1, onto $f:\mathbb{N}\to\mathbb{N}\times\mathbb{N}$)

Can we list elts $\mathbb{N}\times\mathbb{N}$ s.t. each elt of $\mathbb{N}\times\mathbb{N}$ appears exactly once

$(a_0,b_0), (a_1,b_1), (a_2,b_2), \dots$

$f(0)$ $\quad$ ~~$f(1)$~~ $\quad f(2)$ $\quad \dots$

$\mathbb{N}\times\mathbb{N}\times\mathbb{N}$

Proof:

(0,0)	(0,1)	(0,2)	(0,3)	...
(1,0)	(1,1)	(1,2)	(1,3)	...
(2,0)	(2,1)	(2,2)	(2,3)	...

- (4,3) is on list in pos
$f(?)=(4,3)$

$(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), \dots$

$(2,0)$ $(2,1)$ $(2,2)$ $(2,3)$ ...

- $(4,3)$ is on list
  in pos
  $f(?) = (4,3)$

$(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), \dots$
$f(0)$ $f(1)$ $f(2)$ 3 4 5

THM : $E^{\text{meaning}} = \{x \in \mathbb{N} \mid x \text{ is even}\}$ is infinite $\sqsubseteq$
Proof : $f(n) = 2n$ is 1-1 and onto

THM: For any finite set X, X^* is

DEF: Let X (alphabet) be a set. X^* means the set of finite strings whose chars are in X.

$\{0,1\}^* = \{\_, 0, 1, 00, 01, 10, 11, 000, \dots, 111, \dots\}$

THM: $\{0,1\}^*$ is countably infinite

THM: If X is countable and $Y \subseteq X$ then Y is countable

Uncountable Sets

DEF: For real numbers a,b with $a \le b$, (a,b) means $\{x \in \mathbb{R} \mid a < x < b\}$

THM: $(0,1)$ is uncountable

[will set to contradiction]

any set with an infinite subset is infinite

Proof: Suppose $(0,1)$ is countable. Since it is infinite ($\{\frac{1}{n} \mid n \in \mathbb{Z}^+\} \subseteq (0,1)$), then it is countably infini

By def of countably infinite, $\exists$ bijection $f: \mathbb{N} \to (0,1)$

Then write $f(0) = 0.d_{00}\, d_{01}\, d_{02}\, d_{03} \ldots$

$f(1) = 0.d_{10}\, d_{11}\, d_{12}\, d_{13} \ldots$

$f(2) = 0.d_{20}\, d_{21}\, d_{22} \ldots$

$\vdots$

d_{ij} = jth digit in decimal rep. of $f(i)$ (choosing the one that doesn't end in 9999... where th is an issue)

diagonalization argument

need our x to differ from an infinite number of things, but have an infinite number of places to create differences; locations of differences on diagonal)

We find an $x \in (0,1)$ so that $f(n) \neq x$ for all n: (so f is not onto)

$f(0) = 0.500000\ldots$

$f(1) = 0.1415926$

$f(2) = 0.111111$

$\vdots$

$x = 0.112\ldots$

let n^{th} digit of $x = \begin{cases} 1 & \text{if } d_{nn} \neq 1 \\ 2 & \text{otherwise} \end{cases}$

Now x differs in nth digit from $f(n)$ for all n, so $x \neq f(n)$ for all n

$\to x \in (0,1)$ because $\frac{1}{10} \le x < \frac{3}{10}$

So f is not onto $\Rightarrow\Leftarrow$

By contradiction rule, $(0,1)$ is uncountable

THM: $G = \{g \mid g: \mathbb{N} \to \mathbb{N}\}$ is uncountable

not finite b/c has infinite subset (fn that always outputs 0, always outputs 1, $\vdots$)

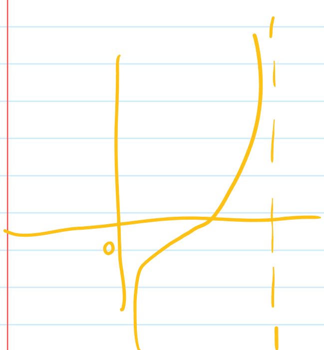
Proof: Suppose ~~$\{g \mid g: \mathbb{N} \to \mathbb{N}\}$~~ is countable; then it's countably infinite

Then $\exists$ 1-1, onto $f: \mathbb{N} \to G$

List elts of G:

$f(0)$:	$f(0)(0)$	$f(0)(1)$	$f(0)(2) \ldots$
$f(1)$:	$f(1)(0)$	$f(1)(1)$	$f(1)(2) \ldots$
$f(2)$:	$f(2)(0)$	$f(2)(1)$	$f(2)(2)$

$\vdots$



Can show f is not onto: there is a fn $h: \mathbb{N} \to \mathbb{N}$

$h(n) = f(n)(n) + 1$

$h \neq f(n)$ for all n

f is not onto $\Rightarrow\Leftarrow$

$h \in G$

Suppose $\exists$ bijection $f: (0,1) \to \mathbb{R}$

then $\mathbb{R}$ is uncountable

THM: $\mathcal{P}(\mathbb{N})$ is uncountable

Proof: Suppose not. Then there is a 1-1, onto $f: \mathbb{N} \to \mathcal{P}(\mathbb{N})$, can construct X s.t. $X \neq f(n)$ for any n:

$$X = \{ n \mid n \notin f(n) \}$$

characteristic fcn

$f(0)$: 0 0 0 0 0 . . . .

$f(1)$: 1 0 1 0 1 0 . . .

$f(2)$: 0 0 1 1 0 1 . . .

For any n, if $n \in f(n)$, then $n \notin X$
if $n \notin f(n)$, then $n \in X$
so $X \neq f(n)$

$\therefore$ f is not onto $\Rightarrow \Leftarrow$

$X = \{ 0, 1, \_, \ldots \}$