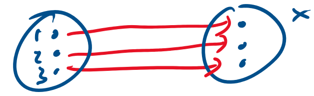
Cardinality



$n(X)=3$

DEF: A set X is finite means $\exists m \in \mathbb{N}$ s.t. there is a bijection $f: \{1,\dots,m\} \to X$

THM: For finite sets X, Y, $n(X) \overset{m}{=} n(Y)$ if and only if there is a bijection $f: X \to Y$

$\Rightarrow$

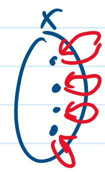
$$\begin{array}{l} \{1,\dots,m\} \xrightarrow{f} X, \quad X \xrightarrow{f^{-1}} \{1,\dots,m\} \\ \{1,\dots,m\} \xrightarrow{g} Y \end{array}$$

$\underset{\text{bijection}}{g} \circ \underset{\text{bijection}}{f^{-1}}$ so comp is bijection

$\Leftarrow$ Pigeonhole

$|X| = |Y|$

DEF: For any sets X, Y, X has the same cardinality as Y means there is a bijection $f: X \to Y$



THM: For any set X, $|X| = |X|$ bijection is identity fxn i_X

For any sets X, Y, $|X| = |Y| \to |Y| = |X|$ suppose $f: X \to Y$ is a bijection then f^{-1} is a bijection $Y \to X$

For any sets X, Y, Z, $|X| = |Y|$ and $|Y| = |Z| \to |X| = |Z|$

bijection $f: X \to Y$ bijection $g: Y \to Z$ $g \circ f$ is a bijection $X \to Z$

Let $\overset{2\mathbb{Z}}{E} = \{x \in \mathbb{Z} \mid x \text{ is even}\}$ $O = \{x \in \mathbb{Z} \mid x \text{ is odd}\}$ $\overset{3\mathbb{Z}}{T} = \{x \in \mathbb{Z} \mid \exists y \in \mathbb{Z} \text{ s.t. } x = 3y\}$

THM: $|E| = |O|$

E	...	−4	−2	0	2	4	6	...
O	...	−3	−1	1	3	5	7	...

let $f(n) = n+1$

Proof: Define $f: E \to O$ by $f(n) = n+1$. Then f is 1-1 and onto

THM: $|2\mathbb{Z}| = |3\mathbb{Z}|$

$2\mathbb{Z}$	...	−4	−2	0	2	4	6	...
$3\mathbb{Z}$	...	−6	−3	0	3	6	9	...

Proof: Define $f: 2\mathbb{Z} \to 3\mathbb{Z}$ by $f(n) = \frac{3}{2} n$ then f is 1-1 and onto

Proof :

DEF: Let X be a set. X^* means the set of finite strings whose chars are in X.

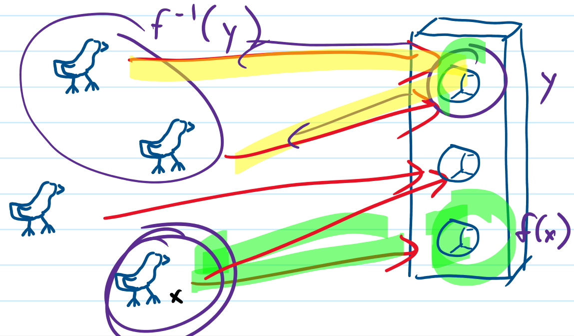
$\{0,1\}^* =$

THM: $\{0,1\}^*$ is

THM: For any finite set X, X^* is

number of elements in Y

THM (Pigeonhole Principle): For any finite sets and with $n(X) > n(Y)$ and any function $f: X \to Y$, f is not one-to-one



$f^{-1}(y) = \{x \mid f(x) = y\}$

$x \in f^{-1}(f(x))$

$f(x') = f(x)$

Proof: Let X and Y be finite sets with $n(X) > n(Y)$ $=m$ and let $f: X \to Y$ [want f is not one-to-one]

Suppose f is 1-1

Write $Y = \{y_1, \ldots, y_m\}$

Let $X_i = f^{-1}(\{y_i\})$ for $1 \le i \le m$

$x \in f^{-1}(y_i)$
$f(x) = y_i$

Now $X = \bigcup_{i=1}^{m} X_i$

$\subseteq$: let $x \in X$ Find y_i s.t. $f(x) = y_i$. Now $x \in X_i$

$\supseteq$: let $x \in \bigcup_{i=1}^{m} X_i$ Find i s.t. $x \in X_i = f^{-1}(\{y_i\})$

$x \in f^{-1}(y_i)$ $x \in X$ by def f^{-1} $x \in f^{-1}(y_j)$

For proof of generalized: $n(X_i) \le k$, so $n(X) \le k \cdot n(Y)$ contradicting $n(X) > k \cdot n(Y)$

For $i \ne j$, $X_i \cap X_j = \emptyset$

Suppose not: $x \in X_i$ and $x \in X_j$ for $i \ne j$
then $x \in f^{-1}(\{y_i\})$ and $x \in f^{-1}(\{y_j\})$
$f(x) = y_i$ $f(x) = y_j$
$x_1, x_2 \in f^{-1}(y_i)$ $\Rightarrow\Leftarrow$

$n(X_i) \le 1$ for all i

suppose not: $\exists X_i$ s.t. $x_1, x_2 \in X_i$ and $x_1 \ne x_2$
then $f(x_1) = y_i$
$f(x_2) = y_i$
f not 1-1 $\Rightarrow\Leftarrow$

split X into nonoverlapping pieces $\to n(X) =$ sum of n(each piece)

$n(X) = \sum_{i=1}^{m} n(X_i) \le \sum_{i=1}^{m} 1 = m = n(Y)$ $\Rightarrow\Leftarrow$

$k \in \mathbb{Z}$

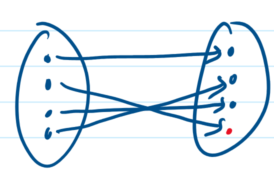
THM (Generalized Pigeonhole Principle): If X, Y are finite sets such that $n(X) > k \cdot n(Y)$ and $f: X \to Y$ then there is a $y \in Y$ s.t. $n(f^{-1}(y)) \ge k+1$

$37 > 3 \cdot 12$ f: students $\to$ months $f(s)$ = month of s's birthday

$37 > 9 \cdot 4$ g: students $\to$ lecture halls at least 10 students w/ same response

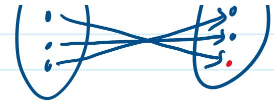
THM: Let X, Y be finite sets with $n(X) = n(Y)$ and let $f: X \to Y$. Then f is 1-1 if and only if f is onto

Proof: Suppose X, Y are finite sets, $n(X) = n(Y)$, and $f: X \to Y$.



$\Rightarrow$: Suppose f is 1-1 but not onto.

Proof: Suppose X, Y are finite sets, $n(X) = n(Y)$, and $f: X \to Y$.



$\Rightarrow$: Suppose f is 1-1 but not onto.

Then $\exists y \in Y$ s.t. $f^{-1}(y) = \emptyset$; let $S = \{y \mid f^{-1}(y) = \emptyset\}$ $\quad n(S) \geq 1$

Now $Y = \{f(x_1)\} \cup \{f(x_2)\} \cup \dots \cup \{f(x_m)\} \cup S$ where all terms are disjoint

$$\begin{aligned} \text{So } n(Y) &= \sum_{i=1}^{m} n(\{f(x_i)\}) + n(S) \\ &= \sum_{i=1}^{m} 1 + n(S) \\ &= m + n(S) \geq m+1 > m \Rightarrow\Leftarrow \end{aligned}$$

$\Leftarrow$: Suppose f is onto but not 1-1

Then $X = \bigcup_{i=1}^{m} f^{-1}(y_i)$ where each $f^{-1}(y_i)$ is non-empty

and $n(X) = \sum_{i=1}^{m} n(f^{-1}(y_i))$

$m > m \quad \Rightarrow\Leftarrow$

THM: If X, Y are finite sets with $n(X) < n(Y)$ and $f: X \to Y$, then f is not onto