Cartesian Product

ordered pair

$A \times B = \{ (x,y) \mid x \in A \wedge y \in B \}$

$(a_1, b_1) = (a_2, b_2)$

means $a_1 = a_2$ and $b_1 = b_2$

$A = \{ a, b, c \}$ $B = \{ 1, 2 \}$

$A \times B = \{ (a,1), (a,2), (b,1), (b,2), (c,1), (c,2) \}$

$B \times A = \{ (1,a), (1,b), (1,c), (2,a), (2,b), (2,c) \}$

$A \times B \times C = \{ (x,y,z) \mid x \in A \wedge y \in B \wedge z \in C \}$ $C = \{ z \}$

$A \times B \times C = \{ (a,1,z), (a,2,z), (b,1,z), (b,2,z), (c,1,z), (c,2,z) \}$

$A \times (B \times C) = \{ (a, (1,z)), \ldots \}$

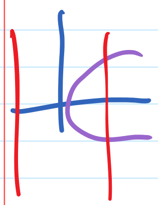
Functions

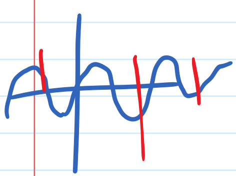
$f = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$

Function : $f : X \to Y$ (domain X, co-domain Y) relates elements of X to elements of Y so that

1) f relates each elt of domain with ≥ 1 thing in co-domain
2) f relates each elt of domain with ≤ 1 thing in co-domain

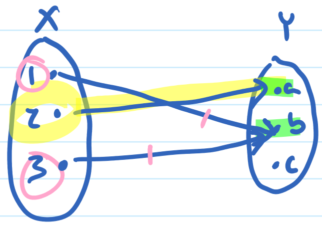
exactly one thing





"f of x"

$f(x)$: the elt of Y that f associates x with



$f(1) = b$
$f(2) = a$
$f(3) = b$

range $(f) = \{a, b\}$

image of $\{1, 3\} = \{b\}$

inverse image of $a = \{2\}$
of $b = \{1, 3\}$
of $c = \emptyset$

range of f: $\{y \in Y \mid \exists x \in X \ f(x) = y\}$

image of A ($A \subseteq X$, $f(A)$): $\{y \in Y \mid \exists x \in A \ f(x) = y\}$

inverse image of y ($y \in Y$): $\{x \in X \mid f(x) = y\}$

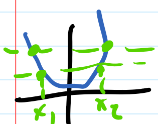
Identity function: $f : X \to X$ defined by $f(x) = x$ $\quad I_X(x) = x$

For fxns $f : X \to Y$ and $g : X \to Y$, $f = g$ means $\forall x \in X,\ f(x) = g(x)$

One-to-one (injective): $f: X \to Y$ is one-to-one if and only if

$$\forall x_1, x_2 \in X \quad x_1 \neq x_2 \to f(x_1) \neq f(x_2)$$

3   -3   T   F



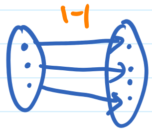
$f: \mathbb{R} \to \mathbb{R}$
$f(x) = x^2$
1-1? NO
$f(3) = 9$
$f(-3) = 9$

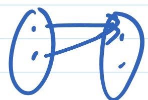
$g: \mathbb{R}^{\text{nonneg}} \to \mathbb{R}$
$g(x) = x^2$
1-1? YES

$h: \mathbb{R} \to \mathbb{R}$
$h(x) = \frac{1}{2}x + 3$
1-1?

at most 1 intersection for each horiz line?

1-1





not 1-1

Suppose $x_1, x_2 \in X$ with $x_1 \neq x_2$ [want: $h(x_1) \neq h(x_2)$]

Suppose bwoc $h(x_1) = h(x_2)$

then $\cancel{\tfrac{1}{2}}x_1 + \cancel{3} = \cancel{\tfrac{1}{2}}x_2 + \cancel{3}$

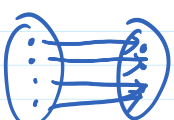
and $x_1 = x_2 \Rightarrow\Leftarrow$

$\therefore h(x_1) \neq h(x_2)$

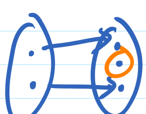
Onto (surjective): $f: X \to Y$ is onto if and only if

$$\forall y \in Y \ \exists x \in X \ f(x) = y$$

$f(X) = Y$



onto



not onto

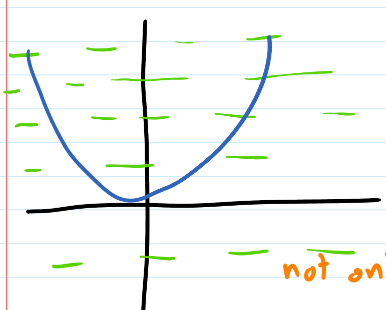
$h: \mathbb{R} \to \mathbb{R}$
$h(x) = \frac{1}{2}x + 3$

onto: Suppose $y \in \mathbb{R}$ [want: $\exists x \in \mathbb{R}$ s.t. $h(x) = y$]
let $x = 2y - 6$

need $\frac{1}{2}x + 3 = y$
$x = 2y - 6$

then $h(x) = \frac{1}{2}(2y-6) + 3$
$= y$

so $\exists x \in \mathbb{R}$ s.t. $h(x) = y$

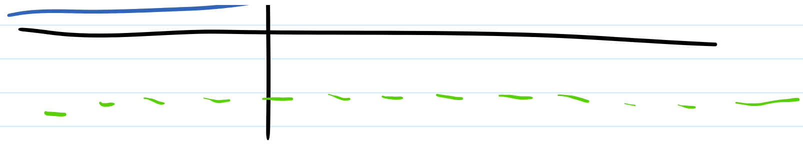


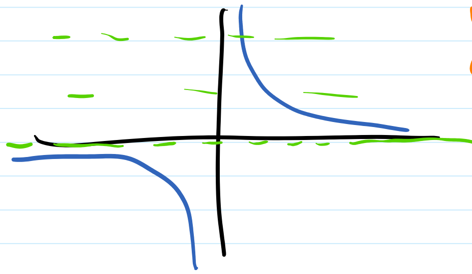
not onto   no x s.t. $x^2 = -4$

does each horiz line intersect at least once

1-1? yes
onto? no

$y = e^x$



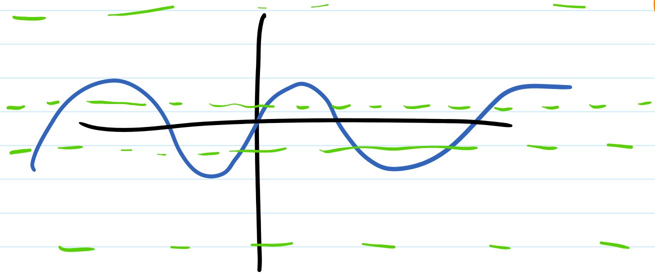


1-1 ? yes

onto? no

$g : \mathbb{R} - \{0\} \to \mathbb{R}$

$g(x) = \frac{1}{x}$



not onto

not 1-1

Inverse Functions

bijective (or f is a 1-1 correspondence) or f is a bijection

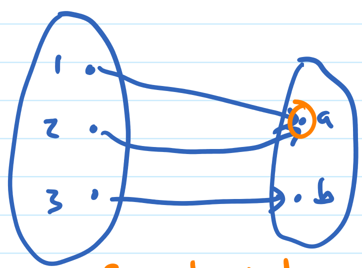
Inverse function: Let $f: X \to Y$ be $\underbrace{\text{1-1 and onto}}_{}$ then $f^{-1}: Y \to X$ defined by $f^{-1}(y) =$ the unique x s.t. $f(x) = y$

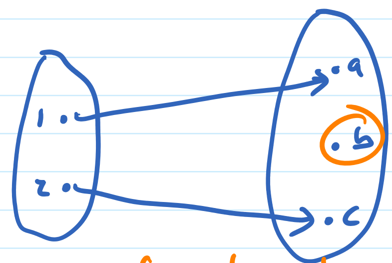
$h: \mathbb{R} \to \mathbb{R}$
$h(x) = \frac{1}{2}x + 3$

$h^{-1}(y) = x$ s.t. $h(x) = y$

$\frac{1}{2}x + 3 = y$

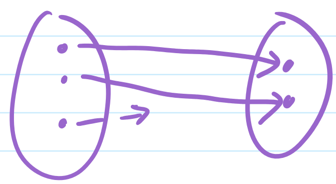
$x = 2y - 6$

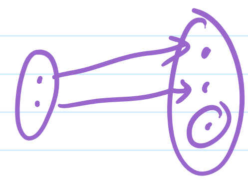




f not 1-1
what would $f^{-1}(a)$

f not onto
what is $f^{-1}(b)$??





Composition of Functions

composition of f with g

Let $f: X \to Y'$ and $g: Y \to Z$ where $Y' \subseteq Y$ then $g \circ f: X \to Z$

is defined by $g \circ f(x) = g(f(x))$

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = 5x-2$

$g: \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$

$g \circ f: \mathbb{R} \to \mathbb{R}$

$$(g \circ f)(x) = g(f(x)) = g(5x-2) = (5x-2)^2$$

$f \circ f^{-1} = I_Y$

$f^{-1} \circ f = I_X$

$$(f \circ g)(x) = f(g(x)) = f(x^2) = 5x^2 - 2$$

THM: Let $Y' \subseteq Y$, $f: X \to Y'$ and $g: Y \to Z$, and f, g both 1-1 then $g \circ f$ is one-to-one

Proof: Suppose $x_1, x_2 \in X$ with $x_1 \neq x_2$ [want $(g \circ f)(x_1) \neq (g \circ f)(x_2)$]

Suppose $(g \circ f)(x_1) = (g \circ f)(x_2)$

then $g(f(x_1)) = g(f(x_2))$

and $f(x_1) = f(x_2)$ b/c g is 1-1

so $x_1 = x_2 \Rightarrow \Leftarrow$ b/c f is 1-1

$\therefore (g \circ f)(x_1) \neq (g \circ f)(x_2)$

THM: Suppose $f: X \to Y$ and $g: Y \to Z$, and f, g both onto then $g \circ f$ is onto

Proof: Let $z \in Z$. [want $x \in X$ s.t. $(g \circ f)(x) = z$]

Then find $y \in Y$ s.t. $g(y) = z$ (can do b/c g onto)

and find $x \in X$ s.t. $f(x) = y$ (can do b/c f onto)

Now $(g \circ f)(x) = g(f(x)) = g(y) = z$