Sets

Set : a mathematical object that contains elements

$x \in S$ (is an element of; S: set) — x is an element of S

$A = \{4, 5, 8, 20, 22, 33, 42\}$

$4 \in A \quad 8 \in A \quad 19 \notin A$ (not an element of)

$B = \{8, 33, 42, 5, 4, 33, 20, 22\}$

$4 \in B \quad 8 \in B \quad 19 \notin B$

enumerate

$C = \{c_1, c_2, \dots, c_n\}$

$x \in C \longleftrightarrow x = c_1 \vee x = c_2 \vee \cdots \vee x \in c_n$

set builder; such that; predicate

$\{x \mid x < 20 \text{ and } x \text{ is prime}\}$

$= \{2, 3, 5, 7, 11, 13, 17, 19\}$

$S = \{x \in D \mid P(x)\}$ $\quad x \in S \longleftrightarrow x \in D \wedge P(x)$

$X = \{1, 2, 4, \dots\}$ $\quad Y = \{x \in \mathbb{Z} \mid x > 0\} = \mathbb{Z}^+$

$Z = \{x \in \mathbb{Z} \mid \exists z \in \mathbb{Z} \text{ s.t. } z \geq 0 \wedge x = 2^z\}$

$2 \in Z$? $\exists z \text{ s.t. } z \geq 0 \wedge 2 = 2^z$? yes (namely $z = 1$)

$G = \{2^y \mid y \in \mathbb{Z} \wedge y \geq 0\}$

$\{f(y) \mid P(y)\}$

$\equiv$

$\{x \mid \exists y \text{ s.t. } P(y) \wedge x = f(y)\}$

$128 \in G$? $\exists y \text{ s.t. } y \in \mathbb{Z} \wedge y \geq 0 \wedge 128 = 2^y$

yes ($y = 7$)

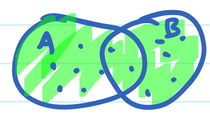
empty set

$\{\} \equiv \emptyset$ $\quad x \in \{\}$ false for every x

$= \emptyset$

Operations

Venn diagram

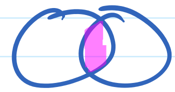


union

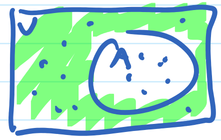
$A \cup B = \{x \mid x \in A \vee x \in B\}$

intersection

$A \cap B = \{x \mid x \in A \wedge x \in B\}$



complement $A^c = \{x \in U \mid x \notin A\}$



$A = \{7\}$ $A^c = \{\ldots$

set difference $A - B = \{x \mid x \in A \wedge x \notin B\}$

$A = \{1, 2, 3\}$ $B = \{x \in \mathbb{Z} \mid 0 \leq x \leq 12 \wedge 3 \mid x\} = \{0, 3, 6, 9, 12\}$

$U = \mathbb{Z}$

$A \cup B = \{0, 1, 2, 3, 6, 9, 12\}$

$A \cap B = \{3\}$

$A - B = \{1, 2\}$

$B - A = \{0, 6, 9, 12\}$

$A^c = \{x \in \mathbb{Z} \mid x < 1 \vee x > 3\}$

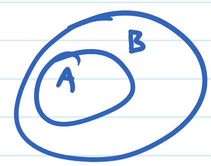
$B^c = \{x \in \mathbb{Z} \mid \sim (0 \leq x \leq 12 \wedge 3 \mid x)\}$

$A \subset B$ means $A \subseteq B \wedge A \neq B$

proper subset of

subset of

$A \subseteq B$ means $\forall x \in U,\ x \in A \rightarrow x \in B$



$A = B$ means $\forall x \in U,\ x \in A \leftrightarrow x \in B$

$\equiv$

$\forall x \in U,\ x \in A \rightarrow x \in B \ \wedge\ \forall x \in U,\ x \in B \rightarrow x \in A$

$\equiv$

$A \subseteq B \ \wedge\ B \subseteq A$

$\forall x, x \in A \rightarrow x \in B$

subset of is transitive

THM: For all sets A, B, C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

Proof: Suppose A, B, C are sets and $A \subseteq B$ and $B \subseteq C$ [want $A \subseteq C$]

$\forall x, x \in A \rightarrow x \in C$

Suppose further that $x \in A$ [want $x \in C$]

Then $x \in B$ (def $\subseteq$)

and $x \in C$ (def $\subseteq$)

distributive

THM: For all sets A, B, C, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof: $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Proof: $\underline{A\cup(B\cap C)\subseteq(A\cup B)\cap(A\cup C)}$

Suppose $x\in A\cup(B\cap C)$ [want $x\in(A\cup B)\cap(A\cup C)$]

Then $x\in A$ or $x\in B\cap C$ (def $\cup$)

2 cases $x\in A$: Then $x\in A\vee x\in B$ and $x\in A\cup B$ (def $\cup$)

also $x\in A\vee x\in C$ and $x\in A\cup C$ (def $\cup$)

So $x\in A\cup B\wedge x\in A\cup C$

and $x\in(A\cup B)\cap(A\cup C)$ (def $\cap$)

element argument

$x\in B\cap C$: Then $x\in B\wedge x\in C$ (def $\cup$)

so $x\in A\vee x\in B$ and $x\in A\vee x\in C$

and $x\in A\cup B$ and $x\in A\cup C$

so $x\in(A\cup B)\cap(A\cup C)$

$x\in(A\cup B)\cap(A\cup C)$ in both cases

$\underline{(A\cup B)\cap(A\cup C)\subseteq A\cup(B\cap C)}$

Suppose $x\in(A\cup B)\cap(A\cup C)$ [want $x\in A\cup(B\cap C)$]

Then $x\in A\cup B$ and $x\in A\cup C$ (def $\cap$)

so $x\in A\vee x\in B$ and $x\in A\vee x\in C$ (def $\cup$)

Two cases: $x\in A$: then $x\in A\vee x\in B\cap C$ (disjunctive add.)

and $x\in A\cup(B\cap C)$ (def $\cup$)

$x\notin A$: then $x\in B$ and $x\in C$ (elimination)

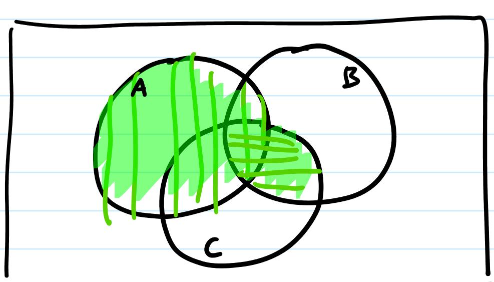
so $x\in B\cap C$ (def $\cap$)

and $x\in A\vee x\in B\cap C$ (disj. add.)

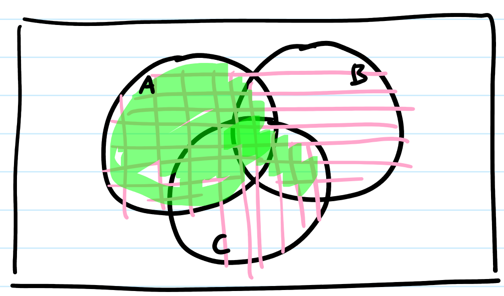
so $x\in A\cup(B\cap C)$ (def $\cup$)

Got $x\in A\cup(B\cap C)$ in both cases

$A\cup(B\cap C)$



$(A\cup B)\cap(A\cup C)$



$x\in A\cup(B\cap C)$ means $x\in A\vee\overbrace{(x\in B\wedge x\in C)}^{x\in B\cap C}$

$\equiv (x\in A\vee x\in B)\wedge(x\in A\vee x\in C)$

$x\in A\cup B \quad\wedge\quad x\in A\cup C$

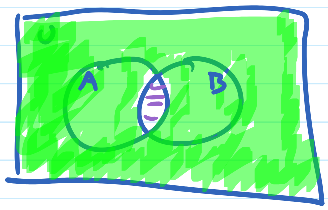
$x\in(A\cup B)\cap(A\cup C)$

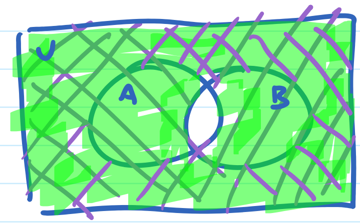
Each logical equivalence has a corresponding set identity

$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$

$$(A \cup B)^c = A^c \cap B^c$$

$$\rightarrow (A \cap B)^c = A^c \cup B^c$$





Power Set

$\mathcal{P}(S) = \{ X \subseteq U \mid \underline{X \subseteq S} \}$ $\quad X \in \mathcal{P}(S) \leftrightarrow X \subseteq S$

power set

$A \subseteq A \quad \forall x, \underline{x \in A \to x \in A}$

$\{3\} \subseteq \{1,2,3\}$

$\mathcal{P}(\{a\}) = \{ \emptyset, \{a\} \}$ $\quad \emptyset \subseteq A \quad \forall x, x \in \emptyset \to x \in A$

$\mathcal{P}(\{1,2,3\}) = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,3,2\} \}$

$\mathcal{P}(\emptyset) = \{\emptyset\}$ $\quad \cancel{\times}\ \emptyset$ $\quad \emptyset \subseteq \emptyset \quad \forall x, x \in \emptyset \to x \in \emptyset$

For all $n \in \mathbb{N}$ and all sets A, if A contains exactly n elements, then $\mathcal{P}(A)$ contains exactly 2^n elements

THM: For all sets A, B, if $A \subseteq B$ then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

Proof: Suppose A, B are sets s.t. $A \subseteq B$ [want $\mathcal{P}(A) \subseteq \mathcal{P}(B)$]

Suppose $X \in \mathcal{P}(A)$ [want $X \in \mathcal{P}(B)$]

Then $X \subseteq A$ (def $\mathcal{P}$)

and $X \subseteq B$ ($\subseteq$ is transitive)

so $X \in \mathcal{P}(B)$ (def $\mathcal{P}$)